Progressions in Primes, III

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$\|f\|_{U^3} \geq \delta$, $f\colon \mathbb{Z}/N\mathbb{Z} \to [-1,1]$

This could happen for e.g. bracket quadratic phases $f(x) = e^{2\pi i n\sqrt{2}\lfloor n\sqrt{3}\rfloor}$.

Theorem: (Inverse theorem for the U^3-norm) If $\|f\|_{U^3} \geq \delta$ then there is a bracket quadratic phase

$q(x) = \sum_{j=1}^{k} c_j x\{x\theta_j\} +$ linear terms

such that $\left|\mathbb{E}_{x\in\mathbb{Z}/N\mathbb{Z}} f(x)e^{-2\pi i q(x)}\right| \gg_\delta 1$.

Here $k = O\delta(1)$.

Theorem: (Inverse theorem for the U^3-norm, II) If $\|f\|_{U^3} \geq \delta$ then there is a 2-step nilsequence $n \mapsto F(g^n x)$ with complexity $U_\delta(1)$ such that $\left|\mathbb{E}_{n\in\mathbb{Z}/N\mathbb{Z}} f(n)F(g^n x)\right| \gg_\delta 1$.

($\|\cdot\|_{U^3}$ large $\Rightarrow$ correlate with a nilsequence.)

G– 2-step nilpotent Lie group

$$\begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ & 1 & \mathbb{R} \\ & & 1 \end{pmatrix} \tag{0.1}$$

Γ– discrete cocompact subgroup

$$\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix} \tag{0.2}$$

G/Γ– 3-dimensional manifold, "2-step nilmanifold"

If $g \in G$ then g acts on G/Γ by left multiplication. $F\colon G/\Gamma \to [-1,1]$ is a "nice" function. $F(g^n x)$ is called a 2-step nilsequence (evaluate on orbit of F).

Exercise:

$$G/\Gamma \longleftrightarrow \{\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} : 0 \leq x, y, z < 1\}$$

$$g = \begin{pmatrix} 1 & \alpha & \beta \\ & 1 & \gamma \\ & & 1 \end{pmatrix} \quad x = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

g^n as an element of G is

$$\begin{pmatrix} 1 & \alpha n & n\beta + \frac{1}{2}n(n-1)\alpha\gamma \\ & 1 & \gamma n \\ & & 1 \end{pmatrix}$$

as an element of G/Γ this is

$$\begin{pmatrix} 1 & \{\alpha n\} & \{n\beta + \frac{1}{2}n(n-1)\alpha\gamma - \alpha n\lfloor \gamma n\rfloor\} \\ & 1 & \{\gamma n\} \\ & & 1 \end{pmatrix}$$

Consider the function $F\colon G/\Gamma \to [-1,1]$ defined by

$$F\left(\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}\right) = e^{2\pi i y}$$

Then $F(g^n x)$ is a bracket quadratic phase.

Conjecture: (Inverse conjecture for the U^{s+1}-norm; II) If $\|f\|_{U^{s+1}} \geq \delta$ then there is a s-step nilsequence $n \mapsto F(g^n x)$ with complexity $O_\delta(1)$ such that $\left|\mathbb{E}_{n\in\mathbb{Z}/N\mathbb{Z}} f(n)F(g^n x)\right| \gg_\delta 1$.

Progressions of Primes: Let's try to model the strategy we used for 3-term progressions
$T_4(\Lambda,\Lambda,\Lambda,\Lambda) = 1+$ fifteen terms, e.g. $T_4(\Lambda-1,\Lambda-1,\Lambda-1,\Lambda-1)$
($T_4(\Lambda,\Lambda,\Lambda,\Lambda) \approx$ 4-term AP of primes)
(Recall we had the "w-trick" so we assume the "primes" are well distributed mod 2,3,etc....)

Now that $T_4(\Lambda-1,\ldots,\Lambda-1)$ is small:

(1) If $|T_4(\Lambda-1,\ldots,\Lambda-1)| \geq \delta$ then (*)
$$\|\Lambda-1\|_{U^3} \geq \delta$$
(Generalized von Neumann)
(2) Inverse conjecture for the U^3-norm $\Rightarrow \left|\mathbb{E}_{n\in[N]}(\Lambda(n)-1)F(g^n x)\right| \gg_\delta 1$
(bit of an issue– $\Lambda-1$ is not bounded)
(3) Derive a contradiction using Vinogradov/Vaughan techniques.

How do we handle 2? GI(s) $\Rightarrow$ Relative GI(s)

Proposition: Suppose that $f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{R}$ is such that $|f(x)| \leq \nu(x)$, a pseudorandom measure. Assume GI(s) the inverse conjecture for the U^{s+1}-norm (true when s=2,3). Then the GI(s) conjecture is true for f too.

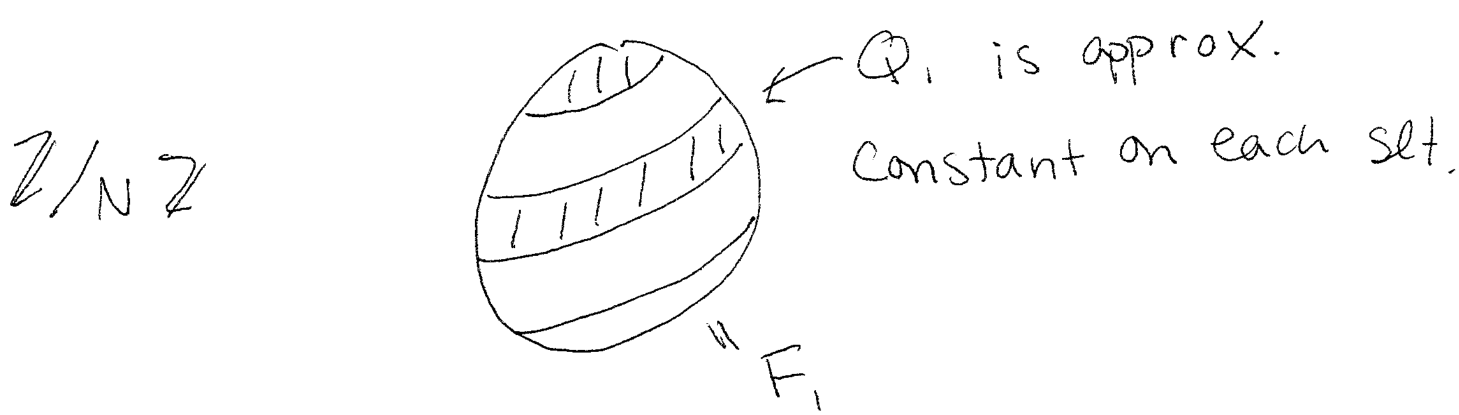
i.e. if $\|f\|_{U^{s+1}} \geq \delta$ then f correlates with an s-step nilsequence.

Historically, we need to contradict (*), i.e. $|\mathbb{E}_{n\in N}(\Lambda(n)-1)F(g^n x)| = O(1)$.

We couldn't rule out the possibility that $\Lambda - 1$ does correlate with a nilsequence

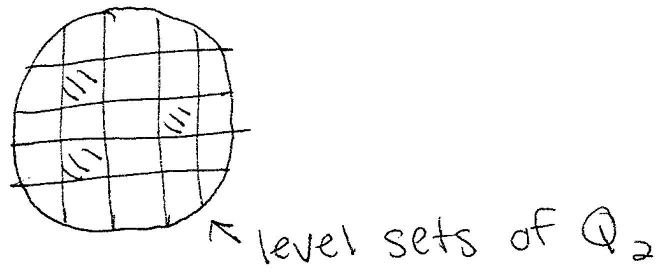
$$\Lambda = 1 + (\Lambda - 1) = F_0 + f_0.$$
If $\|f_1\|_{U^3} = O(1)$ then happy. If $\|f_0\|_{U^3} \geq \delta$ then $|\langle f_0, Q_1\rangle| \gg_\delta 1$ for some 2-step nilsequence Q.

Replace F_0 by "the projection of Λ onto the σ-algebra defined by Q".



Average Λ over each level set.
Write $\lambda = F_1 + f_1$

If $\|f_1\|_{U^3} = O(1)$ STOP.
Otherwise there is Q_2 a 2-step nilsequence, such that $|\langle f_1, Q_1 \rangle| \gg_\delta 1$.



$\Lambda = F_2 + f_2$
(Projection onto σ-algebra defined by Q_1, Q_2).

Claim: The algorithm terminates in finite time because $\|F_j\|_2$ increases at each step. Using the fact that Λ is bounded by the pseudorandom measure ν one can place a global bound on this "energy".

When the algorithm stops, we have $\Lambda = F_k + f_k$ where $\|f_k\|_{U^3} = O(1)$. There, F_k is a projection of Λ onto the σ-algebra generated by $Q_1, \ldots, Q_k$, 2-step nilsequences. In particular (using ν again), F_k is bounded. Thus Λ =(bounded)+(error small in U^3).

Sketch of Proof of Prop: Write f =(bounded)+(error small in U^3). If $\|f\|_{U^3}$ is large then $\|\text{bounded}\|_{U^3}$ is large. So (bounded) correlates with 2-step nilsequences, hence so does f.

One can bypass the need for GI(s) by noting that the iteration I just described didn't use many specific properties of 2-step nilsequences $\mathcal{Q}$.

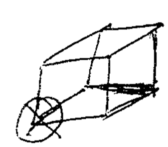
Used:

(1) (inverse conjecture): $\|f\|_{U^3} \geq \delta \Rightarrow \exists\, q \in \mathcal{Q}$ correlating with f.
(2) (converse) If $\|f\|_{U^3} = O(1) \Rightarrow \nexists\, q \in \mathcal{Q}$ correlating with f.
(3) $\mathcal{Q}$ is "an algebra".

(1.) and (2.) are easily seen to be satisfied by a much softer class

$$\|f\|_{U^2}^8 = \langle f, Df \rangle$$

$$Df = \mathbb{E}_{h_1,h_2,h_3} f(x+h_1) \cdot \ldots \cdot f(x+h_1+h_2+h_3)$$



Q=Polynomials generated by Df.

Proof: That the primes contain arbitrarily long APs

$$\Lambda = F + f$$

F bounded
f small in Gowers norm
$F \longleftrightarrow$ set with true density

$$T_4(\Lambda, \Lambda, \Lambda, \Lambda) \approx T_4(F, F, F, F) \gg 0$$

by Szemerédi's Theorem.