Correspondence principle and finitary ergodic theory, III

Terence Tao

29 August 2008

7.) Correspondence Principle for dense subsets of sparse random sets of integers

8.) Correspondence principle for dense subsets of $\mathbb{Z}$

Theorem 1: $\forall\, k \geq 2$, $\forall\, 0 < \delta < 1$, $\exists\, N = N(k, \delta)$ s.t. if $A \subseteq \{1, \dots, N\}$, $|A| > \delta N$
$\Rightarrow A$ contains a k-term AP.

Theorem 2: $\forall\, k \geq 2$, $0 < \delta < 1$, if A a random stationary set of integers
$\mathbb{P}(0 \in A) > 0 \Rightarrow \mathbb{P}(A$ has k-term AP$) > 0$ ($\Rightarrow \mathbb{P}(n \in A) > 0$)

Theorem 2 $\Rightarrow$ Theorem 1: Compactness + contradiction
Thm. 2 true, Thm. 1 false
$\exists\, k, \delta$, $N_n \to \infty$, $A_n \subseteq \{1, \dots, N_n\}$, $|A_n| > \delta N_n$
s.t. none of A_n contain k-AP.

$B_n = A_n - h$, h uniformly chosen at random from $1, \dots, N_n$
(Pass to subseq) $B_n \to B$ (B random set of $\mathbb{Z}$)

$$\mathbb{P}(3, 5 \in B_n, 17 \ni B_n) \to \mathbb{P}(3, 5 \in B, 17 \ni B)$$

$$n + 17 \approx_d h \text{ (approx in density)}$$

B_n "approx stationary" $\Rightarrow B$ is stationary.

7.) Corr...
(Relative Szemerédi Theorem)

Th. 1′: $\forall\, k \geq 2$, $0 < \delta < 1$ $\exists\, N$ s.t. whenever $R \subseteq \{1, \dots, N\}$, $|R| \simeq \frac{N}{\log N}$ and R is sufficiently pseudorandom and $A \subset R$, $|A| \geq \delta\, |R| \geq \frac{\delta N}{\log N}$
$\Rightarrow A$ contains k-AP

Th. 2′ $\Rightarrow$ Th. 1′: compactness + contradiction
Suppose Th. 1′ fails. $\exists\, k, \delta$, $N_n \to \infty$, N_n prime
$R_n \leq N_n$, $|R_n| = \frac{N_n}{\log N_n}$ "increasingly pseudorandom"
$A_n \subseteq R_n$, $|A| \geq \delta\, |R_n|$, A_n has no k-term AP

Random subsets A of $\mathbb{Z}$
$\Updownarrow$
(Borel) prob. measures μ on $2^{\mathbb{Z}}$
$\Updownarrow$ (Carathéodory ext.)
collection of cylinder probabilities

$$0 \leq \mathbb{P}(3, 5 \in B, 17 \ni B) = \mu(c(3, 5, 17)) \leq 1$$

$$c(3, 5, 17) = \{A \subseteq \mathbb{Z} \colon 3, 5 \in A, 17 \neq A\} \subseteq 2^{\mathbb{Z}}$$

obeying compatibility cond.

$$\mathbb{P}(3, 5 \in A) = \mathbb{P}(3, 5 \in A, 17 \ni A) + \mathbb{P}(3, 5 \in A, 17 \in A)$$

$$\mu(c(3, 5)) = \mu(c(3, 5, 17)) + \mu(c(3, 5, 17))$$

virtual prob. measure
a collection of numbers

$$0 \le \mu(c(3,5,17)) \le 1, \text{ etc.}$$

no cond.

dense case
$A_n \to B_n$, μ_n prob. measure on $2^{\mathbb{Z}}$, approx. stationary, stat. in limit
relative case $A_n \to \mu_n$ virtual prob. meas.
approx stationary
approx obey compact cond.

$$\mu(c(3,5)) = \mu_n(c(3,5,17)) + \mu_n(c(3,5,17)) + o(1)$$

$A_n \subseteq R_n$

$$\mu_n(c(3,5,17)) = \mathbb{P}(3,5 \in B_n, 17 \in R_n \backslash B_n) \cdot \log^3 N$$
$$= \mathbb{E}_{n,r}\Lambda(n+3r)\Lambda(n+5r)(y-\Lambda)(n+17r)$$
$$B_n = (A_n - h) \cdot r$$
$$S_n = (R_n - h) \cdot r$$

$$\mu_n(c(3,5)) = \mu_n(c(3,5,17)+\ldots) + o(1)$$
$$o(1) \approx \mathbb{E}_{n,r}\Lambda(n+3r)\Lambda(n+5r)(y-\Lambda)(n+17)$$

8.) Correspondence Principle for convergence of ergodic averages

Mean ergodic thm: $(X,\mu) \circlearrowleft$ prob. space, $f \in L^2(X,\mu)$

$$S_N f = \frac{1}{N}\sum_{n=1}^{N} T^n f, \quad T^n f = f \circ T^{-n}$$

$S_N f$ converges in $L^2(X,\mu)$.

$x_1, x_2, \ldots \in [0,1]$

$$x_n \text{ converges}$$
$$\Updownarrow$$
$$x_n \text{ is cauchy}$$
$$(\forall\, \epsilon\ \exists\, N \text{ s.t. } |x_n - x_m| \le \epsilon,\ n,m \ge N)$$

$$x_n \text{ does not converge}$$
$$\Updownarrow$$
$$x_n \text{ is not cauchy}$$
$$\Updownarrow$$
$$\exists\ \epsilon > 0 \text{ s.t. } \forall\ n > 0\ \exists \text{ arb. large } n \text{ s.t. } |x_n - x_m| \geq \epsilon.$$
$$\Updownarrow$$
$$\exists\ \epsilon \text{ s.t. } \forall\ F_0 \colon \mathbb{Z} \to \mathbb{Z},\ \forall\ n,\ \exists\ m \geq F_0(n) \text{ s.t. } |x_n - x_m| \geq \epsilon.$$
$$\Updownarrow$$
$$\exists\ \epsilon \text{ s.t. } \forall\ F_0 \colon \mathbb{Z} \to \mathbb{Z},\ \exists\ F \colon \mathbb{Z} \to \mathbb{Z} \text{ s.t. } \left|x_n - x_{F(n)}\right| \geq \epsilon\ \forall\ n.$$

$$x_n \text{ converges}$$
$$\Updownarrow$$
$$\forall\ \epsilon\ \exists\ F_0 \colon \mathbb{Z} \to \mathbb{Z} \text{ s.t. } \forall\ F \colon \mathbb{Z} \to \mathbb{Z},\ F > F_0,\ \exists\ N = N(\epsilon, F_0, F), \text{ and } \exists\ n \leq N \text{ s.t. }$$
$$\left|x_n - x_{F(n)}\right| \leq \epsilon.$$

<u>Mean Erg. Thm.:</u>
$\Updownarrow$
<u>Finitary mean erg. thm.:</u> $\forall\ (X, \mu, T),\ f \in L^2(x),\ \|f\|_{L^2(X)} = 1$
$\forall\ \epsilon > 0\ \exists\ F_0 \colon \mathbb{Z} \to \mathbb{Z}$
$\forall\ F > F_0,\ \exists\ N$ s.t. $\left\|S_N f - S_{F(N)} f\right\|_{L^2(X)} \leq \epsilon$ for some $1 \leq n \leq N$.

Suppose F in erg. thm. failed.
$\exists\ (X, \mu, T),\ \|f\|_{L^2(X)} \leq 1,\ \epsilon > 0$ s.t. $\exists$ arb. rapid F, arb. large N s.t. $\left\|S_n f - S_{F(N)} f\right\|_{L^2(X)} > \epsilon$ for some n.

<u>Lemma:</u> $\|S_n S_m - S_m\|_{Op} \leq 2^{(\frac{n}{m})}$
$m \geq F(n)$
$S_n S_m \approx S_m + o(\frac{n}{F(n)})$
$S_n S_m = S_m$

Energy increasing argument:

$$N_1 \subset N_2 \subset N_3 \subset N_4 \subset \ldots \subset N_k$$
$$1,\ F(1),\ F(F(1)), \ldots\ldots, F^k(1)$$

need $\exists\ j$ s.t. $\left\|S_{N_j} f - S_{N_{j-1}} f\right\| < \epsilon$

$$0 \leq \left\|S_{N_j} f\right\|^2_{L^2(X)} = \left\|S_{N_j} f\right\|^2_{L^2} + \underbrace{\left\|S_{N_j} f - SN_{j+1} f\right\|^2_{L^2}}_{\leq \epsilon^2}$$

$$S_{N_j} - S_{N_{j-1}} = S_{N_{j+1}}$$
$$S_{N_{j+1}} f \perp S_{N_{j+1}} f - S_{N_j} f$$