An Inverse theorem for the uniformity seminorms associated with the action of $\mathbb{F}_p^\infty$

Tamar Ziegler

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V. Bergelson
T. Tao

Sz: If $E \subset \mathbb{Z}$ of $d^*(E) > \delta > 0$ then E contains arbitrarily long arithmetic progressions.

$f\colon \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, $\Delta_n f = f(x+n)f(x)$

$$\|f\|_{U_k} = \mathbb{E}_{n\in\mathbb{Z}/N\mathbb{Z}} \|\Delta_n f\|_{U_{k-1}}^{2^k}$$

$$= \mathbb{E}_{x,n_1,\ldots,n_k} f(x)\overline{f}(x+h_1)\cdot\ldots\cdot\overline{f}(x+n_k)f(x+n_1+n_2)\cdot\ldots\cdot f(n+n_1+\ldots+n_k)$$

$f(x) = e^{2\pi i x/N}$, $\|f\|_{U_2} = 1$

$f(x) = e^{2\pi i x^2/N}$, $\|f\|_{U_3} = 1$ but small U_2 norm

Given a function f, what can be said about f if $\|f\|_{U_k} > \delta > 0$?

Inverse conjecture for the Gowers norms: If f has large U_k norm then f correlates with a $k-1$ step nilsequence.

$$\|f\|_{U_3} \text{ large} \longleftrightarrow e^{2\pi i x\alpha\{x\beta\}}$$

Motivation: for this conjecture:

$\mathbf{X} = (X, \mathcal{B}, \mu, T)$ an ergodic m.p.s.

HKG norm:

$$\|f\|_{U_k(\mathbf{X})}^{2^k} = \lim_{n\to\infty} \mathbb{E}_n \|\Delta_n f\|_{U_{k-1}}^{2^{k-1}}$$

$$\|f\|_{U_1(\mathbf{X})} = \lim_{n\to\infty} |\mathbb{E}_n f(T^n x)|$$

HK thm: If $\|f\|_{U_k}^{2^k} > 0$ then f correlates with a function comming from a nilsystem: $\exists$ $g\colon g = \pi^*\tilde{g}$, $\tilde{g}\colon N/\Gamma \to \mathbb{C}$, $\pi\colon X \to N/\Gamma$

$$\langle f, g\rangle > 0$$

Replace the group $\mathbb{Z}/N\mathbb{Z}$ by $\mathbb{F}_p^n$, define similar norms $f\colon \mathbb{F}_p^n \to \mathbb{C}$, $\forall\ n \in \mathbb{F}_p^n$: $\Delta_k f(x) = f(x+h)\overline{f}(x)$

$$\|f\|_{U_k}^{2^k} = \mathbb{E}_{n\in\mathbb{F}_p^n} \|\Delta_n f\|_{U_{k-1}}^{2^k}$$

$$f(x) = e^{2\pi i r x/p},\ \|f\|_{U_2} = 1$$

$$f(x) = e^{2\pi i\langle x,x\rangle/p},\ \|f\|_{U_3} = 1$$

$\|f\|_{U^2}$ small

Suppose $f\colon \mathbb{F}_p^n \to \mathbb{C}$, $\|f\|_{U_k} > \delta$. What can you say?

$k=2$:
$k=3$: Green Tao– p odd, Samordnitsky– p odd

If f has large U_3 then f correlates with a polynomial, where a polynomial of degree $< k$ is a function $\Delta_{h_k} \dots \Delta_{h_1} P = 1$ for every $h_1, \dots, h_k$.

Inverse Conjecture for the Gowers U_3 norm in finite fields: If $\|f\|_{U_k}$ is large then f correlates with a polynomial P of degree $< k$.

Let $\mathbf{X} = (X, \mathcal{B}, \mu, (T_g)_{g \in \mathbb{F}_p^\omega})$, $G = \mathbb{F}_p^\omega$.

Define seminorms

$$\|f\|_{U_{k-1}(\mathbf{X})}^{2^k} = \lim_{n \to \infty} \mathbb{E}_{g \in \Phi_n} \|\Delta_g f\|_{U_{k-1}}^{2^k}$$

$$\|f\|_{U_1(\mathbf{X})} = \lim_{n \to \infty} |\mathbb{E}_{g \in \Phi_n} f(T_g x)| = \left| \int f \right|$$

where Φ_n is a Fölner sequence.

What is a polynomial on an ergodic measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$?

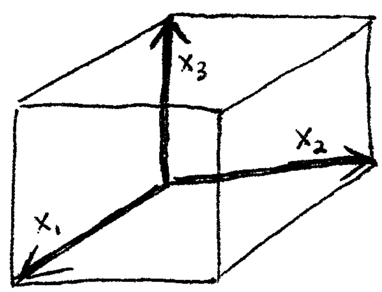
Option 1: Differentiate using group elements $\Delta_k(f) = f(T_h x)\overline{f}(x)$.

A polynomial of degree $< k$
$\Delta_{h_1} \Delta_{h_2} \dots \Delta_{h_k} f = 1$ a.e. $\forall\, h_1, \dots, h_k \in G$.
If $k = 1$: $\Delta_h f = 1 \Leftrightarrow f(T_h x) = f(x)\ \forall\, h \in G \Leftrightarrow f$ is constant a.e.
If $k = 2$: $\Delta_{h_1} \Delta_{h_2} f = 1 \Leftrightarrow \Delta_{h_2} f$ is constant $\Leftrightarrow \Delta_{h_2} f = c(h_1) \Leftrightarrow f(T_{h_2} x) = c(h_1) f(x)$ (f is an eigenfunction).
This way you get $k-1$ order eigenfunctions.

A cubic complex: A sequence of sets

$$X_k \Rightarrow X_{k-1} \Rightarrow X_{k-2} \Rightarrow \dots \Rightarrow X_0$$

each "$\Rightarrow$" stands for $2k$ boundary maps denoted ∂_j^ϵ, $j = 1, \dots, k$, $\epsilon = \pm$, satisfying $\partial_j^{\epsilon_1} \partial_i^{\epsilon_2} = \partial_{i-1}^{\epsilon_2} \partial_j^{\epsilon_1}$.



A finite type invariant of degree $< k$

$$(d^k f)(\overline{x}) = \prod_{\overline{\epsilon} \in \{-1,1\}^k} C^{\epsilon_1 + \dots + \epsilon_k} f(\partial_1^{\epsilon_1} \dots \partial_k^{\epsilon_k}(\overline{x})) = 1, \quad \overline{x} \in X_k$$

Example: (exercise) $X_k = \{\phi \colon I^k = [-1,1]^k \to \mathbb{R}\}$

$$\partial_1^{\pm} \phi = \phi|_{\{x_i = \pm 1\}}$$

then f is a polynomial for degree $< k$ iff $\delta^k f = 0$ for any $\phi \in X_k$.

$(X, \mathcal{B}, \mu, G = (T_g)_{g \in G})$ an ergodic m.p.s.
define a cubic complex of m.p.s. from this system. (generalizing the HK construction for G in stead of $\mathbb{Z}$)
and $\partial_j^{\pm}$ $j = 1, \ldots, k$.

A polynomial of degree $< k$

$$\overline{x} \in (X^{[k]}, \mu^{[k]}, \underbrace{T \times \ldots \times T}_{2^k})$$

$d^k f(\overline{x}) = 1$ a.e. $x \in \mu^{[k]}$

easy lemma: Show that for ergodic measure preserving systems, the 2 definitions are the same.

Thm: Let $(X, \mathcal{B}, \mu, (T_g)_{g \in \mathbb{F}_p^\omega})$ be an ergodic m.p.s. and suppose $\|f\|_{U_k} > 0$. Then f correlates with a function of finite type (polynomial) of degree $< k$.

Thm: (Char. factor version)

$$\|f\|_{U_k} = \langle f, Df \rangle \,, \;\; D(f_1, \ldots, f_{2^k - 1})$$

The universal characteristic factor for D if the factor generated by functions of finite type of degree $\leq k$

$$D_k(f_1, \ldots, f_{2^k-1}) = D(\pi_* f_1, \ldots, \pi_* f_{2^k-1})$$

$G = \mathbb{F}_p^\omega$
G-cocycle: $f \colon \mathbb{F}_p^\omega \times X \to S^1$
$f_{g+h}(x) = f_g(T_h x) f_h(x)$

Main Proposition: $[d^k f_g] = 0$, d^k a cocycle in $(X^{[k]}, \mu^{[k]}, T \times \ldots \times T)$

$$d^k f_g(\overline{x}) = \frac{F(T_g^{[k]} \overline{x})}{F(\overline{x})}$$

then $f \sim f'$, $d^k f = 0$ $\mu^{[k]}$-a.e.