Sept 2 2008 9:00 - 10:00am Michael Taylor

Pseudo-differential Operators on Singular Spaces I

I ΨDOS with Smooth Symbols on Euclidean spaces

II. ΨDOS with rough symbols and paradifferential operators

III. Singular integral operators on Lipschitz Domains, and other Uniformly Rectifiable domains

IV. ΨDOS ~~with~~ on manifolds with bdd geometry

NOTES: http://www.math.unc.edu/faculty/met.

$$p(x,D) = (2\pi)^{-n/2} \int p(x,\xi)\, \hat{u}(\xi)\, e^{ix\xi}\, d\xi$$

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int u(x)\, e^{-ix\cdot\xi}\, dx$$

$$u(x) = (2\pi)^{-n/2} \int \hat{u}(\xi) e^{ix\cdot\xi}\, d\xi$$

$$D_j = \frac{1}{i} \partial/\partial x_j$$

$$p(x,\xi) \in S^m_{\rho,\delta}, \quad 0 \le \rho, \delta \le 1$$

$$\Leftrightarrow \cancel{\left|\frac{\partial^\alpha}{\partial \xi^\alpha} p(x,\xi)\right| \le C\langle\xi\rangle^{m-\rho|\alpha|}}$$

$$\left| D^\alpha_x D^\beta_\xi\, p(x,\xi) \right| \le C \langle\xi\rangle^{m-\rho|\alpha| + \delta|\beta|}$$

L^2 - estimate of $P \in OP\, S^{0}_{\rho,\delta}$ $0 \le \delta < \rho \le 1$

1) Suppose $p \in OP\, S^{-k}_{\rho,\delta} \Rightarrow P : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ $1 \le p \le \infty$

2) Suppose $p \in OP\, S^{-\varepsilon}_{\rho,\delta} \Rightarrow p^* p \in OP\, S^{-2\varepsilon}_{\rho,\delta}$

$\Rightarrow (p^* p)^{N} \in OP\, S^{-2\varepsilon N}_{\rho,\delta}$

$\Rightarrow P : L^2 \to L^2$

3) $p \in OP\, S^{0}_{\rho,\delta}$

$p(x, D) = p^* p \in OP\, S^{0}_{\rho,\delta}$

$|p(x,\xi)| \le M - b$, $b > 0$

$A(x,\xi) = (M - \mathrm{Re}\, p(x,\xi))^{\frac{1}{2}} \in S^{0}_{\rho,\delta}$

$A^* A = M - p(x,D) + r(x,D)$ $r \in OP\, S^{-(\rho-\delta)}_{\rho,\delta}$

$M\|u\|^2_{L^2} - \|pu\|^2_{L^2} = \|Au\|^2_{L^2} - (r(x,D)u, u)$

$\ge -c\|u\|^2_{L^2}$

$\|pu\|^2_{L^2} \le (M + c)\|u\|^2_{L^2}$

<u>Ca - Zyg Thm</u>:

$Pu(x) = \int_{\mathbb{R}^m} k(x,y)\, u(y)\, dy$

Assume $|k(x,y)| \le \dfrac{c}{|x-y|^n}$

$|D_{x,y} k(x,y)| \le \dfrac{c}{|x-y|^{n-1}}$

If $p : L^2 \to L^2$. Then $p : L^p \to L^p$. $1 < p < \infty$

$p(x,D)f = \int k(x, x-y) f(y)\, dy$

where

$$k(x, x-y) = (2\pi)^{-n} \int p(x,\xi) e^{i(x-y)\cdot\xi}\, d\xi$$

$$k(x,z) = (2\pi)^{-n} \int p(x,\xi) e^{iz\cdot\xi}\, d\xi$$

$$z^\alpha k(x,z) = (2\pi)^{-n} \int D_\xi^\alpha p(x,\xi) e^{iz\cdot\xi}\, d\xi$$

$$D_z^\gamma D_x^\beta z^\alpha k(x,z) = (2\pi)^{-n} \int \xi^\gamma D_x^\beta D_\xi^\alpha p(x,\xi)\, e^{+iz\cdot\xi} d\xi$$
$$\in S_{\rho,\delta}^{m-\rho|\alpha|+\delta|\beta|+|\gamma|}$$

if $p(x,\xi) \in S_{\rho,\delta}^m \Rightarrow |D_x^\beta D_y^\gamma k(x, x-y)| \le C_{\beta,\gamma} |x-y|^{-n-m-|\beta|-|\gamma|}$

$$A u(x) = (2\pi)^{-n} \int\int a(x,y,\xi) e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi$$

$$k(x,y) = (2\pi)^{-n} \int a(x,y,\xi) e^{i(x-y)\cdot\xi} d\xi$$

$$p(x,\xi) = (2\pi)^{-n} \int\int a(x,y,\eta) e^{i(x-y)\cdot(\xi-\eta)}\, dy\, d\eta$$
$$= e^{iD_\xi D_y} a(x,y,\xi)\Big|_{y=x}$$

Ex:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u \\ u(0) = f \end{cases}$$

Take Partial Fourier Tran.

$$\frac{\partial \hat{u}}{\partial t} = -|\xi|^2 \cdot \hat{u}$$

Work for $p \in OpS^0_{1,\delta}$, $0 \leq \delta < 1$

Applies to $k(x,y)$ taking value in $\mathcal{L}(H_1, H_2)$
H_1, H_2 Hilbert space

Ex: $H_1 = \mathbb{C}$, $H_2 = \ell^2$
Little Wood - ~~Taylor~~ Paley Partician of Unity
$1 = \sum_{j=1}^{\infty} \psi_j(\xi)^2$
supported on shells with $\xi > \sim 2^j$

$\Phi f = (\psi_0(D) f, \psi_1(D) f, \ldots)$

$\Phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \ell^2)$
$\Phi^* (g_0, \ldots) = \sum \psi_j(D) g_j$

$\Phi^* \Phi = I$
C.Z. $\Rightarrow$ $\Phi : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, \ell^2)$
$\Phi^* : L^p(\mathbb{R}^n, \ell^2) \to L^p(\mathbb{R}^n)$

$1 < p < \infty$