Sept 2. 1:30 – 2:30. Maciej Zworski
Spectrum & Scattering Theory

1. Spectral theory on Compact Riem. manifolds
2. Scattering on the line
3. Spectral & Scattering theory on surfaces with Cusps
4. Scattering on manifolds with cylindrical ends.

(X, g) Compact C^∞ manifold. $g = \sum_{1\le i,j\le n} g_{ij}\, dx^i dx^j$

$$\Delta_g = -\frac{1}{\sqrt{\bar g(x)}} \sum \partial_{x_j} (g^{ij} \sqrt{\bar g}\ ) \partial_{x_i}$$

$$\bar g = \det g_{ij}, \quad |\xi|_g^2 = \sum g^{ij} \xi_i \xi_j$$

$d\mathrm{vol}_g = \sqrt{\bar g}\, dx$
$\|u\|^2 = \int |u|^2 \sqrt{\bar g}\, dx$

$$H^2 = \{u : \partial^\alpha u \in L^2, \ |\alpha| \le 2\}$$

<u>Fact:</u> $\|u\|_{H^2}^2 \cong \|\Delta u\|_{L^2}^2 + \|u\|_{L^2}^2$ (Elliptic regularity)

<u>Self-adjointness</u>:

$$\langle u, \Delta v\rangle_{L^2} = \langle \Delta u, v\rangle_{L^2}$$

<u>Resolvent set of A</u>: $z \in \mathbb{C}$. s.t. $\boxed{(A-z)^{-1}} : H \to H$ bdd.
Resolvent of A.

The spetrum of A = Complement of resolvant set.

Claim: $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$. then

$\Delta_g - z : H^2 \to L^2$ is invertible

$\operatorname{Im} \langle (\Delta_g - z) u, u \rangle \geq - \operatorname{Im} z \|u\|^2$.

$\|(\Delta_g - z) u\|_{L^2} \geq \cancel{\|u\|_{L^2}} \, |\operatorname{Im} z| \|u\|_{L^2}$

$\|(\Delta_g - z) u\|_{L^2} \geq \|u\|_{H^2} - C\|u\|_{L^2}$

Conclude: $(\Delta_g - z)^{-1} : L^2 \to H^2 \hookrightarrow L^2$
$\operatorname{Im} z = 0$ Rellich, Compact.

Rellich: Spectrum of $(\Delta_g - z)^{-1}$ is discrete & $\to 0$.

$\operatorname{Spec}((\Delta_g - z)^{-1}) = \{(\mu_j - z)^{-1}\}$, $\mu_j \to \infty$, $\mu_j \in \mathbb{R}$

$\Delta_g u = \lambda u$

$\langle \Delta_g u, u \rangle = \lambda \|u\|^2$

$\| $

$\int \|\nabla u\|^2 \geq 0$

$\Rightarrow \lambda \geq 0$

Write: $\mu_j = \lambda_j^2$:

$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$

X connected, so $\lambda_0 = 0$ is simple.

Weyl LAW: $N(\lambda) = \#\{\lambda_j \le \lambda\}$

Theorem: $N(\lambda) = \dfrac{vol(X)\cdot vol(B_{\mathbb{R}^n})}{(2\pi)^n}\lambda^n + E(\lambda)$

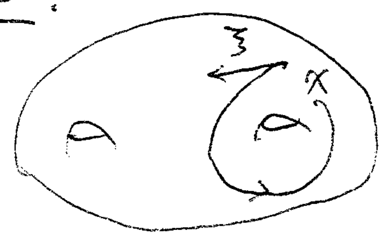
Weyl (1910), $E(\lambda) = o(\lambda^n)$

Levitan 1955 ; $E(\lambda) = O(\lambda^{n-1})$

Avakumović 1956

Hormander 1969:

Dynamics:



Helton's Theorem:

Either $\overline{\{\lambda_i - \lambda_j : i, j\}} = \mathbb{R}$

or the flow Φ^t is periodic

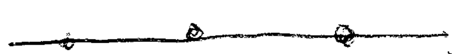
Duistermaat — Guillemin

If $vol_{S^*X}\{p \in S^*X : \exists T(p), \Phi^{T(p)}(p) = p\} = 0$

Then $E(\lambda) = o(\lambda^{n-1})$

Example: S^n $\quad \gamma_k + \frac{(n-1)^2}{4} \oplus (k + \frac{n-2}{2})^2$

multiplicity: $\binom{n+k}{n} - \binom{n+k-2}{k} \sim \frac{2k^{n-1}}{(n-1)!}$



flow is periodic!

Hassel (Now)