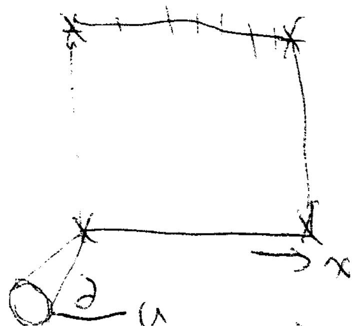
Sept 3, 2008 9:00am – 10:am Michael Taylor

Pseudo-differential operators III

$u \in H^{\sigma,2}$ is Microlocally regular on Conic open $\Gamma \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ $\Leftrightarrow$ $Pu \in C^\infty$ $\forall\, p \in OPS^m_{\rho,\delta}$ with $ES(P) \subset \Gamma$ $\Leftrightarrow$ $Qu \in C^\infty$ for some $Q \in OPS^r_{\rho,\delta}$ elliptic in Γ

$u \in H^{s,p}$ microlocally on Γ if and only if $Pu \in H^{s,p}$ $ES(p) \subset \Gamma$ $\Leftarrow$ $Qu \in H^{s,p}$ for a $Q \in OPS^0_{\rho,\delta}$ elliptic in Γ

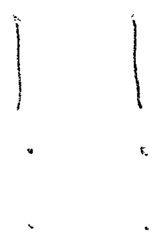


$u = \chi_S \in H^{\frac{1}{2}-\varepsilon,\varepsilon}$

wave front set : normal lines for sides. everything on four vertex

$u \in H^{1-\varepsilon}$ microlocally

$\frac{\partial}{\partial x} u$

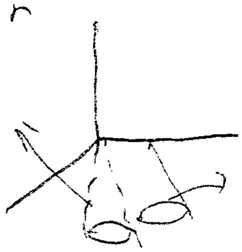


$\frac{\partial^2}{\partial y \partial x} u$ four δ - fums

$\delta \in H^{-1-\varepsilon}(\mathbb{R}^2)$ $\forall \varepsilon > 0$

$u \in \chi_\Theta \in H^{\frac{1}{2}-\varepsilon}$

Microlocally in $H^{\frac{1}{2}-\varepsilon}$



Microlocally in $H^{1-\varepsilon}$

$\frac{\partial}{\partial x} u$:

Surface measure

$\frac{\partial}{\partial y}\frac{\partial}{\partial x}$

Linear measure on z - axis

$\frac{\partial^3}{\partial z \partial y \partial z}$: . δ-function

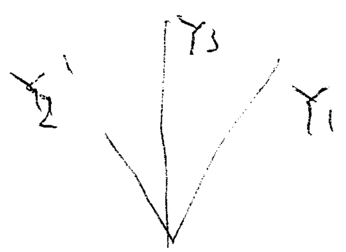
$\delta \in H^{-\frac{3}{2}-\varepsilon}$

$u_j(t,x) = H(t - x \cdot e_j) \quad x \in \mathbb{R}^3$

$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$

$\begin{cases} \Box = \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u_1 = 0 \\ \Box u_2 = 0 \\ \Box u_3 = 0 \end{cases}$

$\Box v = u_1 u_2 u_3 \quad , v(t,x) = 0. \text{ for } t < 0$



$w = Y_1 Y_2 Y_3 v$

$\Box w = Y_1 Y_2 Y_3 (u_1 u_2 u_3) = \delta_0$

$\begin{cases} \cancel{w = 0 \quad t} \\ w = \frac{\delta(t - |x|)}{4\pi t} \quad t > 0 \end{cases}$

$\partial_t^2 u = \Delta u$ $\quad\quad$ Δ is negative.

$$\Lambda = \sqrt{1-\Delta}$$
$$v = \Lambda u$$
$$w = \partial_t u$$
$$\begin{cases} \partial_t v = \Lambda w \\ \partial_t w = \Delta \Lambda^{-1} w \end{cases}$$
$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & \Lambda \\ \Delta\Lambda^{-1} & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

$$\Delta\Lambda^{-1} = -\Lambda \mod OpS^0$$

$$\frac{\partial u}{\partial t} = i A(t, x, D) u \qquad A(t, x, \xi) \in S^1$$
$$A_1(t, x, \xi) \text{ real}$$

Solution Operators $S(t,s)$ by

$S(t,s)\, u(s) = u(t)$

$S(t,s) : H^{s/2}(R^n) \to H^{s/2}(R^n)$

Study $P(t) = S(t,0) P_0 S(0,t)$, $P_0 \in OpS^m_{1,0}$

$S(t,s)\, u(s) = u(t)$

$\partial_t S(t,s) \cancel{u(s)} = i\, A(t) S(t,s)$

$\partial_s S(t,s)\, u(s) + S(t,s)\, u'(s) = 0$

$\partial_s S(t,s) = -i\, S(t,s)\, A(s)$

$$\begin{cases} \dfrac{\partial P}{\partial t} = i A(t)\, P(t) - i\, P(t) A(t) = i[A(t), P(t)] \\ P(0) = P_0 \end{cases}$$

Solve $Q' = i[A(t), Q(t)] + R(t)$ ← smoothing $\quad Q(0) = Q_0$

Claim: $P(t) - Q(t)$ is smoothing

i.e., $S(t,0)P_0 - Q(t)S(t,0)$ smoothing

$u(t) = S(t,0)P_0 f$. $v(t) = Q(t)S(t,0)f$

$\partial_t u = iA(t)u$, $u(0) = P_0 f$

$\partial_t v = iA(t)u + \underbrace{R(t)S(t,0)f}_{g \text{ smooth}}$, $v(0) = P_0 f$

$u(t) - v(t) = \int_0^t S(t,s)g(s)ds$

Try $Q(t) = q(t,x,D) \sim \sum_{j\geq 0} q_j(t,x,D)$

$q_j(t,x,\xi) \in S_{1,0}^{m-j}$

$\partial_t q(t,x,D) = i[A(t), q(t,x,D)] \mod OPS^{-m}$
$\quad\quad\quad\xi$

$\partial_t q(t,x,\xi) \sim H_{A_1} q + \{A_0, q\} + \sum_{|\alpha|\geq 2} \frac{1}{\alpha!} (D_\xi^\alpha A\, D_x^\alpha q - D_\xi^\alpha q\, D_x^\alpha A)$

$\partial_t q_0 = H_{A_1} q_0$

$$\begin{cases} \left(\frac{\partial}{\partial t} - H_{A_1}\right) q_0(t,x,\xi) = 0 \\ q_0(0,x,\xi) = P_0(x,\xi) \end{cases}$$

$$\begin{cases} \partial_t q_1 = H_{A_1} q_1 + b_1(t,x,\xi) \\ q_1(0,x,\xi) = 0 \end{cases}$$

$\partial_t q_j = H_{A_1} q_j + b_j(t,x,\xi)$

$u \in H^{s,2}(R^m)$ $\quad \Sigma = WF(u)$

$WF(S(t,0)u)$

Say Γ is open conic in complement of Σ.

So $ES(p_0) \subset \Gamma$, $P_0 u \in C^\infty$

$S(t,0) P_0 S(0,t) S(t,0) u$, smooth.

So $p(t)$ smooth out $S(t,0)u$

So $C(t)$ flow from time 0 to time t given by

$H_{A(t)}$, $WF(S(t)u) = C(t) WF(u)$

$\partial_t u = X(t) u$ $\qquad$ $X(t)$ smooth vector fields

$OPS^m_{1,0}(M)$ well defined on compact mfd M.