Sept 11:00-12:00 Richard Melrose

Real blow up

- ~~Commutativity~~ Commutation of blow up
- Manifolds with Corners

Manifolds with Corners

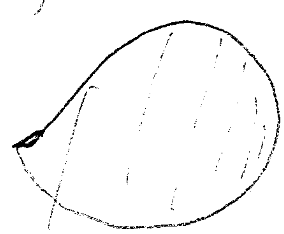
M - manifold covered by coordinate systems & smooth transition

'with Corners - $\mathbb{R}^n \xrightarrow{\text{replace}} [0,\infty)^k \times \mathbb{R}^{n-k}$

- Near each point $p \in M$. (adapted) coordinates

$\underbrace{x_1, \ldots, x_k}_{\geq 0}$, $\cancel{x_{k+1}, \ldots, x_n}$ $y_1, \ldots, y_{n-k}$

Boundary hypersurfaces, locally given by $x_j = 0$



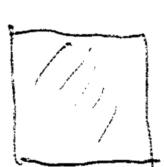
Exclude by demanding hypersurface be embedded

H boundary hypersurface

$= \{\rho_H = 0\}$ $\rho_H \in C^\infty(M)$

$d\rho_H \neq 0$ on H

$\rho_H \geq 0$



good

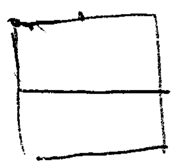
(product) p-submfd

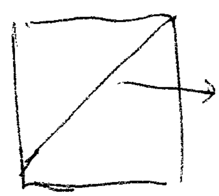
$[M; Y]$

Y closed, embedded

Near each point of Y, $\exists$ adapted coord. on M s.t. $Y = \{x_i = 0,\ 1 \le i \le s,\ y_j = 0,\ 1 \le j \le l\}$

$\Rightarrow$ Collar neighborhood thm.

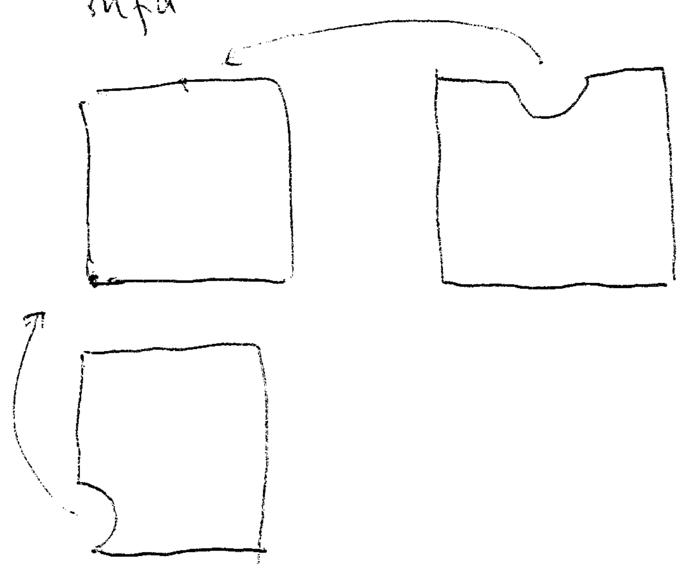




not good

$\mathrm{Diag} = \{x_1 - x_2 = 0\}$

$\beta : [M, Y] \to M$ for Y as a closed p-mfd



Multiple blow ups

$Y_1, Y_2 \subset M$ p-submfds $\qquad$ when is $[[M;Y_1], \overset{\text{lift}}{\tilde{Y}_2}]$

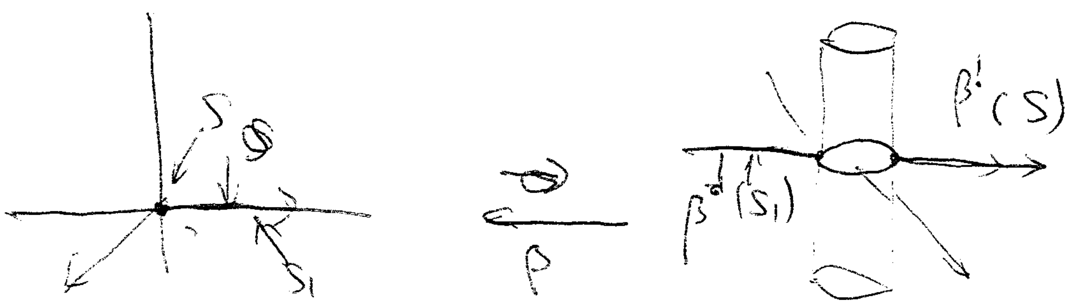
Lift of a submfd $S \subset M$ under blow up of Y:

$\beta : [M, Y] \longrightarrow M$

$$\beta^!(S) = \begin{cases} \cancel{S \subset Y}\ \beta^{-1}(S) & S \subset Y \quad ① \\ \overline{\beta^{-1}(S \setminus Y)} & S \setminus Y \neq \emptyset \quad ② \end{cases}$$

S closed

closed in $[M;Y]$



$Y_1 \subset M$ p-submfd

$Y_2 \subset M$ $\qquad \beta_1 : [M, Y_1] \to M. \quad \beta_2 : [M, Y_2] \to M$

when is $\beta_1^!(Y_2)$ a p-submfd

Y_1, Y_2 jointly p-submfds.

$\Leftrightarrow$ p-submfds. if $p \in Y_1 \cap Y_2$, $\exists$ (adapted) coord. on M s.t.

$Y_i = \{x_t = 0\ t \in I_i \subset \{1, \dots, k\},\ y_s = 0,\ s \in J_i \subset \{1, \dots, n-k\}\}$

$i = 1, 2$, $\Rightarrow$ Bott's clean intersection

$\Rightarrow Y_1 \cap Y_2$ is a p-submfd.

Lemma: If Y_1, Y_2 are jointly p-submfd ~~in the~~
Then $\beta^!(Y_2) \subset [M; Y_1]$ is a p-submfd.

Then $[[M, Y_1], Y_2] \cong [[M, Y_1], \beta_1^!(Y_2)]$
exists

$[[M, Y_2], Y_1]$ exists.
$\underset{\beta_2^!(Y_1)}{\overset{\|}{}}$

iff $Y_1 \subset Y_2$, $Y_2 \subset Y_1$
or Y_1 is transversal to Y_2

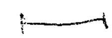
N.B. Complements $\beta_2^{-1}(\beta_1^{-1}(Y_2 \cap Y_1))$
$\cong \beta_1^{-1}(\beta_2^{-1}(Y_1 \cap Y_2))$
are open dense and canonically iso.

$Y_1 \pitchfork Y_2 \iff I_1 \cap I_2 = \phi$, $J_1 \cap J_2 = \phi$ locally
$\iff M = M_1 \times M_2$, $Y_1 = Y_1' \times M_2$ $(Y_1' \subset M_1)$
$Y_2 = M_1 \times Y_2'$ $(Y_2' \subset M_2)$
$[[M, Y_1], Y_2] \cong$ $\cancel{[[M, Y_2], Y_1]}$ $[M_1, Y_1'] \times [M_2; Y_2']$

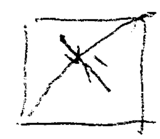
Y_1, Y_2 related or transversal. Commutation of blow up is o.k.
$Y_1 \subset Y_2$, $I_1 \supset I_2$, $J_1 \supset J_2$

Radial vector fields. $R = x_1 \partial_{x_1} + \cdots + y_\ell \frac{\partial}{\partial y_\ell}$ for J_2
$R_2 = R_2 + x_\ell \partial_{x_\ell}$

Non p-submfd.

X = mfd with boundary 

$X^2 \supset \text{Diag} = \{(z,z) \in X^2 : z \in X\}$



Pseudo-differential operators $\Leftrightarrow$ kernels on X^2

C^∞ away from Diag, nice singularities along Diag

Want an intrinsic space of pseudodiff operators

quantize $\longleftrightarrow$ $x\frac{\partial}{\partial x}$ ∂_y, vector fields tangent to the boundary

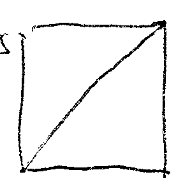
① How to resolve the diagonal $\subset X^2$ to a p-submfd

② How to resolve Lie algebra of tangent vector fields

Find $Y \subset X^2$, so that ① & ② solved by $[X^2, Y]$ (∂X connected)

① $Y \supset \text{Diag}(\partial X)$ all bad points

$\quad = \partial(\text{Diag})$



② Need to have $x\partial_x$ tangent to Y

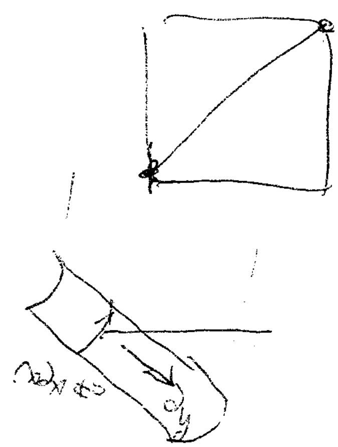
$\partial\text{Diag} = \{(y,y),\ y=y',\ (\cancel{x,y},\ \cancel{x',y'})\} \subset Y$

∂_y tangent X

$\Rightarrow \quad Y = (2X)^2$

$X_D^2 = [X^2 : (2X)^2]$

Resolves both the diagonal and $\mathcal{V}_b = \mathrm{Span}\ x\partial_x, \partial_y$



lift to $\overset{be}{\overline{X}}$ · to diagonal