I. De Rham's cohomology:

a) Definitions: let M^n be a smooth manifold oriented of dimension n; we will note $C^\infty(\Lambda^k T^* M)$ the vector space of smooth differential form of degree k, i.e. $\alpha \in C^\infty(\Lambda^k T^* M)$ is a smooth section of the bundle $\Lambda^k T^* M \to M$ each $\alpha(x) : (T_x M)^k \to \mathbb{R}$ is a alternate k linear form.

In local coordinate $(x_1, \dots x_n)$ we have $\alpha = \sum_{I = \{i_1 < i_2 < \dots < i_k\}} \alpha_I \, dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_{|I| = k} \alpha_I \, dx_I$

where α_I are smooth function. The differential of α is

$$d\alpha = \sum_{|I| = k} d\alpha_I \wedge dx_I$$

This define a linear differential operator $d: C^\infty(\Lambda^k T^* M) \to C^\infty(\Lambda^{k+1} T^* M)$

One of the important properties of d is that its square is zero: $d \circ d = 0$, in particular we have $\operatorname{Im} d \subset \operatorname{Ker} d$, or equivalently if

$Z^k(M) = \{ \alpha \in C^\infty(\Lambda^k T^* M) \,/\, d\alpha = 0 \}$ is the space of closed forms and

$B^k(M) = \{ d\beta ;\ \beta \in C^\infty(\Lambda^{k-1} T^* M) \}$ is the space of exact forms then

$B^k(M) \subset Z^k(M)$

Definition: the $k^{th.}$ groups of de Rham's cohomology is

$$H^k_{dR}(M) = Z^k(M) / B^k(M)$$

It is clearly an invariant of the diffeomorphism type of M, i.e. if M and M' are diffeomorphic by $\varphi : M \to M'$ then the map of "pull back" φ^* induced an isomorphism between $[\varphi^*] : H^k_{dR}(M') \xrightarrow{\sim} H^k_{dR}(M)$

Moreover it has two important properties (amongst a long list of other properties)

b) Two properties:

i) The Poincaré's Lemma: If $\mathbb{B}^n$ is a Euclidean n-ball then

$$H^k_{dR}(\mathbb{B}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \{0\} & \text{if } k > 0 \end{cases}$$

ii) Mayer-Vietoris exact sequence: If $U \cup V = M$, where U and V are open, there are a partition of unity (φ_U, φ_V) and with it we can easily show that

$$\begin{array}{ccccccc} & & & & (\beta, \gamma) & \longrightarrow & (\beta_{|U \cap V} - \gamma_{|U \cap V}) \\ 0 \to C^\infty(\Lambda^k T^*(U \cup V)) & \to & C^\infty(\Lambda^k T^* U) \oplus C^\infty(\Lambda^k T^* V) & \longrightarrow & C^\infty(\Lambda^k T^*(U \cap V)) & \to & 0 \\ \alpha & \to & (\alpha_{|U}, \alpha_{|V}) & & & & \end{array}$$

From the exactness of this short exact sequence, we can deduce the following long exact sequence called Mayer Vietoris exact sequence.

$$\dots \to H^{k-1}_{dR}(U\cap V) \to H^k_{dR}(U\cup V) \to H^k_{dR}(U) \oplus H^k_{dR}(V) \to H^k_{dR}(U\cap V) \to H^{k+1}_{dR}(U\cup V) \to \dots$$

c) <u>De Rham's Theorem</u>: This Mayer-Vietoris exact sequence can be generalized to a open covering with more than two open sets but in this case we do not get an exact sequence but a spectral sequence and these 2 properties with general result in algebraic topology implies: (cf Bott-Tu's book)

Theorem: (de Rham) De Rham's cohomology is isomorphic to cohomology with real coefficient: $H^k_{dR}(M) \simeq H^k(M,\mathbb{R})$

In particular, de Rham's cohomology is an invariant of the homotopy type of M.

<u>II - L^2_{loc} cohomology:</u>

a) <u>Definitions</u>. We consider the vector space $L^2_{loc}(\Lambda^k T^*M)$ of L^2_{loc} section of $\Lambda^k T^*M$ in local coordinate $\alpha \in L^2_{loc}(\Lambda^k T^*M)$ is $\alpha = \sum_{|I|=k} \alpha_I \, dx_I$ with

$\alpha_I \in L^2_{loc}(U)$ (U open set where the coordinate are defined)

we can also define an unbounded operator d using Stokes's formula:

we say that $\alpha \in L^2_{loc}(\Lambda^k T^*M)$ satisfies $d\alpha \in L^2_{loc}(\Lambda^{k+1}T^*M)$ if in any local coordinate $(x_1 \dots x_n)$ $U \to V \subset \mathbb{R}^n$ and for any compact set $K \subset U$.

there is a $C_K > 0$.

$$\forall \varphi \in C^\infty_0(\Lambda^{n-k-1}T^* \overset{\circ}{K}) \qquad \left|\int_U \alpha \wedge d\varphi\right| \leq C_K \sum_{|I|=n-k-1} \int_K |\varphi_I|^2 \, dx_1 \dots dx_n$$

then $d\alpha \in L^2_{loc}$ is defined to be the $\beta \in L^2_{loc}$ such that

$\forall \varphi \in C^\infty_0(\Lambda^{n-k-1} T^*M)$

$$\int_M \alpha \wedge d\varphi = (-1)^{k-1} \int_M \beta \wedge \varphi$$

We also have $d\circ d = 0$, and we can define similarly L^2_{loc} cohomology with

$$Z^k_{2,loc}(M) = \{ \alpha \in L^2_{loc}(\Lambda^k T^*M) \,/\, d\alpha = 0\}$$

$$B^k_{2,loc}(M) = \{ \alpha \in L^2_{loc}(\Lambda^k T^*M) \,/\, \alpha = d\beta,\ \beta \in L^2_{loc}\}$$

Introduce $I\alpha = x \llcorner \alpha$

$$= \sum \alpha_I \,(x \llcorner dx_I).$$

we have $d I \alpha = \sum d\alpha_I \wedge x \llcorner dx_I + \sum \alpha_I \, d(x \llcorner dx_I)$

$$I d\alpha = \sum dx_I(x) \cdot dx_I + \sum d\alpha_I \wedge x \llcorner dx_I$$

$$d(x \llcorner dx_I) = \sum_I x_{i_k} (-1)^{d} dx_{I - i_k}$$

$$= c \; dx_I$$

$d(x \llcorner dx_1) = dx_1 \qquad d\, x \llcorner (dx_1 \wedge dx_2) = d(x_1 dx_2 - x_2 dx_1) = 2\, dx_1 \wedge dx_2$

$$d I \alpha + I d\alpha = \sum d\alpha_I(x) \cdot dx_I + k\,\alpha$$

$\rho_t = \frac{1}{t^n} \rho\left(\frac{x}{t}\right)$

$$\rho_t * \alpha = \int \rho(x-y)\, \alpha_I(ty)\, t^k$$

$$\partial_t (\rho_t * \alpha) = \sum \int \rho(x-y)\, d\alpha_I(ty)\, y \llcorner dx_I \ldots$$

$$= \sum \int \rho \,(d I\alpha + I d\alpha) + k \, \rho * \alpha.$$

$$B\alpha = \sum_I \int_0^1 (\rho_t * \alpha_I \,(I dx_I)\, t^k)\, dt$$

$$\varphi_t \alpha = \frac{1}{t^{n-k}} \sum \int \rho\left(\frac{x}{t}\right) * \alpha_I \; dx_I$$

$$= t^k \sum \int \rho * \alpha_I(ty)\; dx_I$$

$$\partial_t \varphi_t \alpha = k\, t^{k-1} \varphi_t \alpha + t^k \sum \int \rho * d\alpha_I(ty)\, ty \; dx_I$$

$$= k\, t^{k-1} \varphi_t \alpha + t^{k-1} \underbrace{\int \rho_t * d\alpha_I(y)\; dx_I}_{\rho_t * (dI\alpha + I d\alpha) - k \rho_t * \alpha.}$$

$$\boxed{B\alpha = \sum_I \int_0^1 (\rho_t * \alpha_I)\, t^k dt \;\; (x \llcorner dx_I)}$$

$$H^k_{2,loc}(M) = Z^k_{2,loc}(M) / B^k_{2,loc}(M)$$

it is also an invariant of the diffeomorphism type of M, in fact an invariant of the Lipschitz structure of M.

b) Properties:

i) Poincaré's lemma: If $\mathbb{B}^n$ is an Euclidean ball then

$$H^k_{2,loc}(\mathbb{B}^n) = \begin{cases} \{0\} & k>0 \\ \mathbb{R} & k=0 \end{cases}$$

There are several proofs of this fact, one is to show that the usual proof of Poincaré's lemma works into this L^2_{loc} setting, another one is to show a Künneth type formula

$H^k_{2,loc}(I \times U) \simeq H^k_{2,loc}(U)$ if $I \subset \mathbb{R}$ is an open interval and

because $\mathbb{B}^n \simeq]-1,1[^n$ we can reduce this computation to a ~~n=1~~ computation.

I will here given another proof using regularization.

Proof Consider $\alpha \in L^2_{loc}$ such that $d\alpha = 0$ on $\mathbb{B}^n \simeq \mathbb{R}^n$ $\deg \alpha \geq 1$

and take $\rho \in C^\infty_0(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \rho = 1$ and if $\alpha = \sum \alpha_I \, dx_I$

defined $\tilde{\alpha} = \mathcal{S}\alpha = \sum (\rho * \alpha_I) \, dx_I \in C^\infty(\Lambda^k T^* \mathbb{R}^n)$

Step 1: show that $d\tilde{\alpha} = 0$ $(= \mathcal{S} d\alpha)$ hence because $H^k_{dR}(\mathbb{R}^n) = \{0\}$ there is $\beta \in C^\infty$ $\tilde{\alpha} = d\beta$.

Step 2: There is a bounded operator $B : L^2_{loc} \to L^2_{loc}$ such that

$$\alpha - \tilde{\alpha} = d\,B\alpha = dB\alpha \,(+\, B\,d\alpha)$$

Conclusion

$$\alpha = \tilde{\alpha} + dB\alpha = d(\beta + B\alpha) \qquad \square$$

We also have a Mayer-Vietoris exact sequence for L^2_{loc} cohomology, hence

Thm
$$H^k_{2,loc}(M) \simeq H^k(M, \mathbb{R})$$

We can in fact give a direct proof of the induced isomorphism

$$H^k_{2,loc} \simeq H^k_{dR}$$

On any manifold using the above trick, de Rham constructed a sequence of regularization operator

$\mathcal{S}_i : L^2_{loc} \to C^\infty$ and associated $R_i : L^2_{loc} \to L^2_{loc}$

such that $\mathcal{S}_i \alpha \to \alpha$; $\alpha = \mathcal{S}_i \alpha + dB_i\alpha + B_i d\alpha$ if $d\alpha \in L^2_{loc}$

Another important remark: We can also define cohomology with compact support $H^k_c(M) = \{\alpha \in C^\infty_0(\Lambda^k T^*M) / d\alpha = 0\} / d\, C^\infty_0(\Lambda^{k-1} T^*M)$

and we have a nondegenerate pairing

$$H^k_{dR}(M) \times H^{n-k}_c(M) \to \mathbb{R}$$

$$([\alpha],[\beta]) \to \int_M \alpha \wedge \beta.$$

or a nondegenerate pairing $H^k_{L^1_{loc}}(M) \times H^{n-k}_c(M) \to$ which induced Poincaré duality.

This implies that the range of d on L^1_{loc} is closed:

if $d\alpha_i \to \gamma$ on L^1_{loc} then necessary. (1) $d\gamma = 0$

(2) $\int_M \gamma \wedge \beta = 0 \ \forall [\beta] \in H^{n-k-1}_c(M)$

hence γ is zero in L^1_{loc}-cohomology, i.e. $\gamma = d\alpha_\infty$ for some $\alpha_\infty \in L^1_{loc}$.

III - L^2 cohomology and Harmonic forms:

We assume now that M^n is moreover compact without boundary, then the cohomology spaces of M have finite dimension and $L^1_{loc} = L^1$ can be endowed with a structure of Hilbert space with a Riemannian metric g.

$$\|\alpha\|^2_{L^2} = \int_M |\alpha(x)|^2_g \, dvol_g$$

We can define the adjoint operator of d: $d^*: C^\infty(\Lambda^{k+1} T^*M) \to C^\infty(\Lambda^k T^*M)$.

with the rule $\langle d\alpha, \beta\rangle_{L^2} = \langle \alpha, d^*\beta\rangle_{L^2}$, note that d^* depend on g.

Some interesting consequence of de Rham's regularization operator are the following.

1) $C^\infty(\Lambda^k T^*M)$ is dense in $\{\alpha \in L^2 / d\alpha \in L^2\}$ endowed with the graph norm $N(\alpha) = \sqrt{\|\alpha\|^2_{L^2} + \|d\alpha\|^2_{L^2}}$

2) $B^k_{L^2} = \overline{d\, C^\infty(\Lambda^{k-1} T^*M)}$ where the closure is taken with respect to the L^2 structure

Hence if $\mathcal{H}^k(M) = Z^k_{L^2}(M) \cap (B^k_{L^2})^\perp = Z^k_{L^2}(M) \cap (d\, C^\infty(\Lambda^{k-1} T^*M))^\perp$

$$= \{\alpha \in L^2(\Lambda^k T^*M) / d\alpha = 0 \text{ and } d^*\alpha = 0\}$$

Then Theorem (Hodge-de Rham).

$$H^k_{dR}(M) \simeq H^k_{L^2}(M) \simeq \mathcal{H}^k(M)$$

[illegible]

In fact $d+d^*$ is an elliptic operator hence $\mathcal{H}^k(M) \subset C^\infty$.

Moreover if we introduce $\Delta = dd^* + d^*d = (d+d^*)^2$ the Hodge de Rham Laplacian we have $\langle \Delta\alpha, \alpha\rangle = \|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2$ hence

$$\mathcal{H}^k(M) = \{\alpha \in C^\infty(\Lambda^k T^*M) / \Delta\alpha = 0\}$$

We can give another proof of Hodge de Rham theorem using Elliptic theory

We have an orthogonal decomposition (Hodge-de Rham-Kodaira).

$$L^2(\Lambda^k T^*M) = \underbrace{\mathcal{H}^k(M) \oplus \overline{d\, C^\infty(\Lambda^{k-1}T^*M)}}_{\operatorname{Ker} d} \oplus \overline{d^*\, C^\infty(\Lambda^{k+1}T^*M)}$$

We have a Green operator of $d+d^*$: an operator $G : W^{2,k} \to W^{2,k+1}$, $C^\infty \to C^\infty$.

such that for any $\alpha \in C^\infty$

$\alpha = h(\alpha) + dG\alpha + d^*G\alpha$ where $h(\alpha)$ is the orthogonal projection of α on $\mathcal{H}^k(M)$.

This implies the decomposition $C^\infty(\Lambda^k T^*M) = \underbrace{\mathcal{H}^k(M) \oplus d\, C^\infty(\Lambda^{k-1}T^*M)}_{\operatorname{Ker} d} \oplus d^* C^\infty(\Lambda^{k+1}T^*M)$

Hence $H^k_{dR}(M) \simeq \mathcal{H}^k(M)$.

In the remaining two lectures, I will focus on the work of Cheeger on the L^2 cohomology of pseudomanifold and the conclusion of the work of Cheeger-Goresky-MacPherson about L^2 cohomology of stratified space with iterated conical singularity, then if time permits I will discuss the case of manifold with cusps (such as hyperbolic manifold of finite volume).

Sept 3, 2:30 - 3:30 Gilles Carron

Cohomology I

1. An Introduction to Hodge de Rham
2. Case of Pseudo-manifolds (Cheeger)
3. Manifolds with Cusps

I. De Rham Cohomology

(M, g) smooth oriented n-dimensional mfd. $C^\infty(\Lambda^k T^*M)$ = smooth k-forms

In local coordinates $(x_1, \dots, x_n)$

$$\alpha \in C^\infty(\Lambda^k T^*M)$$

$$\alpha = \alpha_{p_1 \cdots p_k} dx_{p_1} \wedge \cdots \wedge dx_{p_k}$$

$$= \sum_{|I| = k} \alpha_I dx_I$$

where $\alpha_I \in C^\infty$

$$d\alpha = \sum d\alpha_I \wedge dx_I \in C^\infty(\Lambda^{k+1} T^*M)$$

$$d : C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$$

$$d^2 = 0 \Rightarrow \operatorname{Im} d \subset \ker d$$

~~Def~~: k^{th} de Rham cohomology space

$$H^k_{dR}(M) = \frac{\{\alpha \in C^\infty(\Lambda^k T^*M) \mid d\alpha = 0\}}{d\, C^\infty(\Lambda^{k-1} T^*M)}$$

$H^k_{dR}(M)$ are invariant of the diffeo type of M.

two properties:

Poincaré Lemma: B^n is a Euclidean ball

$$H^k_{dR}(B^n) = \begin{cases} \mathbb{R} & k=0 \\ 0 & k>0 \end{cases}$$

Meyer - Vietoris Sequence:

$M = U \cup V$, U, V open

with a partition of Unity subject to open cover

$$0 \to C^\infty(\Lambda^k T^* M) \to C^\infty(\Lambda^k T^* U) \oplus C^\infty(\Lambda^k T^* V)$$

$$\alpha \to (\alpha|_U, \alpha|_V)$$

$$\to C^\infty(\Lambda^k T^*(U \cap V)) \to 0$$

We deduce the long exact sequence of M-V:

$$H^k_{dR}(U \cap V) \to H^k_{dR}(M) \to H^k_{dR}(U) \oplus H^k_{dR}(V) \to$$

$$H^k(U \cap V) \to H^{k+1}_{dR}(M)$$

generalizes to more than 2 open sets

~~these two properties~~ Thm (de Rham)

$H^k_{dR}(M) \cong H^k(M; \mathbb{R})$

II. L^2_{loc} Cohomology

a) <u>Definition</u>.

$L^2_{loc}(\Lambda^k T^*M)$ the space of L^2 local sections of $\Lambda^k T^*M$ $\quad \alpha \in L^2_{loc}(\Lambda^k T^*M)$

$$\alpha = \alpha_I dx^I$$

with $\alpha_I \in L^2_{loc}$

d is defined as an unbdd operator with $\alpha \in L^2_{loc}(\Lambda^k T^*M)$ verifying $d\alpha \in L^2_{loc}$

if in any coordinate $(x_1, \dots, x_n)$. $U \to V \subset \mathbb{R}^n$
$\forall K \subset\subset U$, $\exists C_K$. $\forall \varphi \in C_0^\infty(\Lambda^{n-k-1} T^*M)$, supp K

$$\left|\left| \int_U \alpha \wedge d\varphi \right|\right| \leq C_K \sqrt{\int_K |\varphi|^2}$$

$$\varphi = \sum \varphi_I dx_I$$

when $d\alpha$ to be s.t. $\forall \varphi \in C^\infty(\Lambda^{n-k} T^*M)$
Stokes identity holds

$$\int \alpha \wedge d\varphi = (-1)^{k+1} \int d\alpha \wedge \varphi$$

Similarly $\quad d^2 = 0$
and we can define

$$Z^k_{L^2_{loc}}(M) = \{\alpha \in L^2_{loc}(\Lambda^k T^*M) \mid d\alpha = 0\}$$

$$B^k_{L^2_{loc}}(M) = \{d\beta,\ \beta \in L^2_{loc}(\Lambda^{k-1} T^*M),\ d\beta \in L^2_{loc}\}$$

$H^k_{2,loc}(M) = Z^k_{2,loc}(M) \,/\, B^k_{2,loc}(M)$

L^2_{loc} - cohomology of M.

b) Properties.

Poincaré Lemma for L^2_{loc}-cohomology
We show that lemma for $R^n \simeq B^n$
$\alpha \in L^2_{loc}(\Lambda^k T^* R^n)$

$$\alpha = \sum \alpha_I \, dx_I \qquad \alpha_I \in L^2_{loc}(R^n)$$

Take $\rho \in C^\infty_c(R^n)$, $\int_{R^n} \rho = 1$.

Define $\quad \mathcal{J}_\alpha = \rho * \alpha = \sum(\rho * \alpha_I)\, dx_I$
$\in C^\infty(\Lambda^k T^* R^m)$

$$d\mathcal{J}_\alpha = \rho\, d\alpha \quad \text{if } d\alpha \in L^2_{loc}$$

~~there~~ is $B: L^2_{loc} \to L^2_{loc}$
s.t.

$$\alpha = \mathcal{J}_\alpha + dB\alpha + Bd\alpha \qquad \text{if } d\alpha \in L^2_{loc}$$

if $d\alpha = 0 \Rightarrow \cancel{d}\alpha = \mathcal{J}_\alpha \cancel{d\alpha} + dB\alpha$
$k>0 \quad d\mathcal{J}_\alpha = 0 \underset{\text{in } C^\infty}{\overset{}{\underset{\text{P.Lemma}}{\Rightarrow}}} \mathcal{J}_\alpha = d\beta, \ \beta \in C^\infty$

M-V sequence is also exact

L^2_{loc}

Conclusion:

$$H^k_{2,loc}(M) \cong H^k(M; \mathbb{R})$$

In fact, we can even give a direct isomorphism between $H^*_{2,loc} \cong H^*_{dR}$

$\{\rho_i\}$ a sequence of smoothing operators $\alpha \in L^2_{loc}$.

$\rho_i \alpha \in C^\infty$.

$\beta_i : L^2_{loc} \to L^2_{loc}$ s.t.

$$\alpha = \rho_i \alpha + d\beta_i \alpha + \beta_i d\alpha \qquad \text{if } \alpha, d\alpha \in L^2_{loc}$$

<u>Remark:</u> We can define ~~H^k_{dR}~~ $H^k_c(M)$ from $C^\infty_c(\Lambda^k T^*M)$

Poicaré duality:

$$H^{n-k}_c(M) \otimes H^k_{dR}(M) \longrightarrow \mathbb{R}$$

$$([\alpha] \qquad [\beta]) \to \int \alpha \wedge \beta$$

is nondegenerate.

Consequence: the Range of d in L^2_{loc} is closed

Take (β_i) a sequence of L^2_{loc} forms

$$d\beta_i \xrightarrow{L_{loc}} \gamma$$

<u>Remark</u>: $\Delta = dd^* + d^*d = (d+d^*)^2$

$$\langle \Delta\alpha, \alpha \rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2$$

$$\mathcal{H}^k(M) = \{\alpha : \Delta\alpha = 0\}$$

Another more classical proof of Hodge - de Rham theorem with elliptic theory

<u>Starting point</u>: Hodge de Rham - Kodaira.

$$L^2(\Lambda^k) \cong \mathcal{H}^k(M) \oplus \overline{dC^\infty(\Lambda^k)} \oplus \overline{d^*C^\infty(\Lambda^{k+1})}$$

$d + d^*$ is elliptic

~~there~~ is a Green's Operator G. $W^{2,k} \to W^{2,k+1}$

s.t. if $\alpha \in C^\infty(\Lambda^k)$

$$\alpha = h(\alpha) + dG(\alpha) + d^*G(\alpha)$$

$h(\alpha)$ = projections of α onto $\ker(d+d^*)$

If α is closed.

$$d^*G\alpha = 0.$$

Every de Rham cohomology class has a harmonic representative.