Sept4 1:30 - 2:30 Gilles Carron

I: Pseudo-mfd

a) ~~Def~~, A pseudo manifold of dim n is a simplicial complex of ~~dimensional~~ X s.t. &

1) each point lives in n simplex

2) Each $(n-1)$-simplex is a face of exactly 2 n-simplex

3) Each n simplex is compatibility oriented

Ex: 1) Smooth mfd. (because it can be triangulated

2) G is a compact Lie group acting on M, C^∞ mfd. M/G is a pseudo mfd

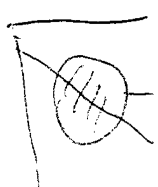
3) If $X^n \subset P^N(\mathbb{C})$ is an algebraic variety. then X is a pseudo mfd.

b) a Geometry on pseudo mfd.

Assume that X^n is endowed with a piece wise continuous Riem. metric

Let $X^n \supset X^{n-1} \supset \cdots \supset X^0$

Let $\Sigma = X^{n-2}$ the ~~simplex~~ singular locus of X^n

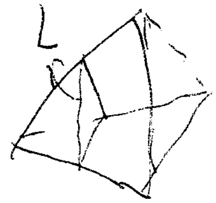


$X_{reg} = X^n - \Sigma$ is the regular part of X

- near each point of X_{reg}. the geometry is the one of $B_\varepsilon^n \subset \mathbb{R}^n$

- If $p \in X^k - X^{k-1}$, p has a neighborhood $U \simeq V \times C(L)$

  V is a Euclidean k-ball. $C(L)$ is a cone over a $(n-k-1)$-pseudomfd called the link at p



the geometry on $C(L_{reg})$ is $(ds)^2 + r^2 h$ $\quad r \in [0, \varepsilon)$

II: L^2 Cohomology of pseudomfds.

Assume X^n is compact.

a) L^2 cohomology on X_{reg}. We have

$$C_0^\infty(\Lambda^k T^* X_{reg})$$

$$\text{and } L^2(\Lambda^k T^* X_{reg}) = L^2(\Lambda^k T^* X)$$

We can define

$$d_k : \mathcal{D}^k \to L^2$$

$$\mathcal{D}^k = \{ \alpha \in L^2(\Lambda^k) \mid d\alpha \in L^2 \}$$

$$= \{ \alpha \in L^2(\Lambda^k) \mid \exists C \quad \forall \varphi \in C_0^\infty(\Lambda^{k+1} T^* X_{reg}) \quad |\langle \alpha, d^* \varphi \rangle| < C \|\varphi\|_{L^2} \}$$

$Z_2^k(X) = \{\alpha \in L^2(\Lambda^k) \mid d\alpha = 0\}$

$B_2^k(X) = d\mathcal{D}^{k-1}$

$L^2 H_2^k(X) = Z_2^k / B_2^k$

<u>Lemma:</u> With de Rham's regularization operator can show that

$$H_2^k(X) = \frac{\{\alpha \in L^2(\Lambda^k T^* X) \cap C^\infty(X_{reg}) : d\alpha = 0\}}{d(\mathcal{D}^{k-1} \cap C^\infty(\Lambda^{k-1} T^* X_{reg}))}$$

b) <u>Hodge theory</u>:

$$Z_2^k(X) = \left(d^* C_0^\infty(\Lambda^{k+1} T^* X_{reg})\right)^\perp$$

$$\mathcal{H}^k(X) = \{\alpha \in L^2(\Lambda^k T^* X) \mid d\alpha = 0 \text{ and } d^*\alpha = 0\}$$
$$= \left(d C_0^\infty(\Lambda^{k-1} T^* X_{reg})\right)^\perp \cap \left(d^* C_0^\infty \Lambda^{k+1} T^* X_{reg}\right)^\perp$$

We get for free the Hodge - de Rham-Kodaira decomposition

$$L^2(\Lambda^k T^* X) = \mathcal{H}^k(X) \oplus \overline{d C_0^\infty(\Lambda^{k-1} T^* X_{reg})}^{L^2}$$
$$\oplus \overline{(d^* \Lambda^{k+1} T^* X_{reg})}^{L^2}$$

$$Z_2^k(X) = \mathcal{H}^k(X) \oplus \overline{d C_0^\infty(\Lambda^{k-1} T^* X_{reg})}$$

$$B_2^k(X) = d\mathcal{D}^{k-1}$$

Two problems for Hodge - de Rham Thm.

$$\mathcal{H}^k(X) \cong H_2^k(X)$$

1) Range of d_{k-1} is not closed

2) ~~$d\mathcal{D}$~~

$$d\mathcal{C}_0^\infty(\Lambda^{k-1}T^*X_{reg}) \subset d\mathcal{D}^{k-1}$$

Introduce minimal extensions of d_{k-1}

$\mathcal{D}^{k-1}_{min}$:= Closure of $C_0^\infty(\Lambda^k T^* X_{reg})$ in

$\mathcal{D}^{k-1}$ for the operator. ~~norm~~ $\alpha \mapsto \|\alpha\|_{L^2} + \|d\alpha\|_{L^2}$

$$dC_0^\infty(\Lambda^{k-1}T^*X_{reg}) \subset d\mathcal{D}^{k-1}_{min}$$

two conditions to have Hodge theorem:

1) $\mathcal{D}^{k-1}_{min} = \mathcal{D}^{k-1}$

2) Range of d_{k-1} is closed

Questions: when these conditions are satisfied?

Lemma: $X^n = \bigcup_{i=1}^N U_i$ open cover, with partition of unity with bounded gradient. If 1) and 2) hold on each U_i, they hold for X^n.

C. Local computation.

1) A finite interval. $I = [a,b]$.
$H_2^0(I) \cong \mathbb{R}$, $H_2^1(I) \equiv \{0\}$, then
the range of d is closed and
$\mathcal{D}_{min} = \mathcal{D}$

2) A product. $I \times Y$
$\pi : I \times Y \to Y$
$\pi^* : L^2(\Lambda^k T^* Y) \longrightarrow L^2(\Lambda^k T^*(I \times Y))$
and an integration map:
$L^2(\Lambda^k T^*(I \times Y)) \longrightarrow L^2(\Lambda^k T^* Y)$

$$dt \wedge \alpha + b \longrightarrow \int_I b$$

1 and 2 hold on Y $\Leftrightarrow$ they hold for $I \times Y$.

$H_2^k(I \times Y) = H_2^k(Y)$

First. $B^k \times C(L)$

3). L^2 cohomology of finite cone $C(L)$
$\alpha \in L^2(\Lambda^k T^* C(L))$
on $C(L_{reg})$. $\alpha = dr \wedge a + b$

$$\|\alpha\|_{L^2}^2 = \int_0^1 \left(|a|^2 r^{-2k+2} + |b|^2 r^{-2k}\right) r^{n-1}\, dr$$

$$d\alpha = dr \wedge (-d_L a + b') + d_L b$$

Two homotopy operator:

$\bar{K}\alpha = \int_{\frac{1}{2}}^{1} \left(\int_{z}^{r} dr \, dz \right) dZ$

$\underline{K}\alpha = \int_{1}^{t} a\,dt$

If $k > \frac{m}{2}$, ~~then~~ K is bdd in L^2

$\|\frac{1}{r} \underline{K}\alpha\|_{L^2} \le \|\alpha\|_{L^2}$ $(*)$

$\deg \alpha = k > \frac{m}{2}$

Moreover, if $\alpha \in \mathcal{D}^k$,

$\alpha = d\underline{K}\alpha + \underline{K}d\alpha$

$H_2^k(C(L)) = \{0\}$ and the range of d_k is closed

Can we approximate α by a sequence α_i with support $(0,1) \times L_{reg}$

$C(L) = [0,1] \times L / \{0\} \times L$

In fact with $(*)$, we can approximate by α_i with support in a $[\frac{1}{i}, 1] \times L$

The other degree more Complicated (especially $k = \frac{n-1}{2}, \frac{n}{2}$)

<u>Final answer</u>

<u>Prop</u>. Assume that on L, $\mathcal{D}_{min} = \mathcal{D} +$ Range d_L is defined, $H_2^{\frac{n-1}{2}}(L) = 0$. then (1) & (2) hold on $C(L)$.

And

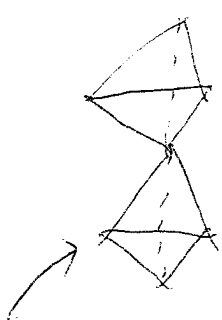
$$IH^k_2(c(L)) = \begin{cases} 0 & \text{if } k \geq \frac{n}{2} \\ IH^k_2(L) & \text{if } k < \frac{n}{2} \end{cases}$$

<u>Conclusion:</u> Assume that X is a pseudomfd with even dimension with only even dim singular strata, then

$$IH^k_2(X) \cong \mathcal{H}^k(X)$$

<u>Rm:</u> A pseudomfd doesn't necessarily satisfy the Poincaré duality.

<u>Ex:</u>



X^2

$H^0(X) \sim \mathbb{R}$

$H^1(X) = \{0\}$

$H^2(X) \cong \mathbb{R}^2$