Sept 5. 9:00-10:00 Maciej Zworski

Surfaces with Cusp ends.

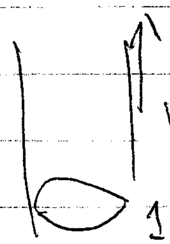
X 2-dimensional, metric g.

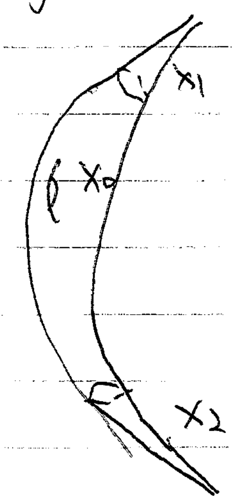
$X = X_0 \cup X_1 \cup X_2$

X_0 is compact. $\partial X_0 = \partial X_1 \cup \partial X_2$

X_j is isometric to $S^1 \times [1, \infty)$

$g|_{X_j} = (dy^2 + a_j^{-2} d\theta^2)/y^2$





$\Delta_g|_{X_j} = -y^2(\partial_y^2 + a_j^2 \partial_\theta^2)$

$d\mathrm{vol}_j = \dfrac{d\theta\, dy}{a_j y^2}$

$P: u \in L^2(X) \longmapsto \left(\int u|_{X_1}(y, \theta)\, d\theta,\ \int u|_{X_2}(y, \theta)\, d\theta \right)$

$\in L^2\left([1,\infty), \frac{dy}{y^2}\right)^2$

$\mathcal{H}_{int} = \ker P \subset L^2(X)$

$L^2(X) = \mathcal{H}_{int} \oplus L^2\left([1,\infty); \frac{dy}{y^2}\right)^2$

$u \in C_c^\infty(X)$, $P\Delta_g u = \begin{pmatrix} -y^2\partial_y^2 & 0 \\ 0 & -y^2\partial_y^2 \end{pmatrix} Pu$

$y = e^r$, $\partial_y = e^{-r}\partial_r$

$-y^2\partial_y^2 = e^{2r}(e^{-r}\partial_r)^2 = -\partial_r^2 + \partial_r$ on $L^2(\mathbb{R}_+, e^{-r}dr)$

$e^{-r/2}(-\partial_r^2 + \partial_r)e^{r/2} = -\partial_r^2 + \frac{1}{4}$ on $L^2(\mathbb{R}_+, dr)$

$e^{\frac{r}{2}}u \in L^2(e^{-r}dr) \Leftrightarrow u \in L^2(dr)$

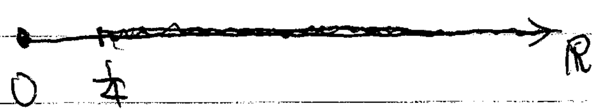
Crucial Fact:

$(\Delta_g - i)^{-1}(I-P)$ is compact

"Proof": $\Delta_g|_{X_1} = \bigoplus_{n\in\mathbb{Z}} y^2(-\partial_y^2 + n^2)_{L^2(\frac{dy}{y^2})}$

$\simeq \bigoplus_{n\in\mathbb{Z}} \left(-\partial_r^2 + \frac{1}{4} + e^{2r}n^2\right)_{L^2(dr)}$

for $n \neq 0$, cpt resolvant!



Spec(Δ_g)

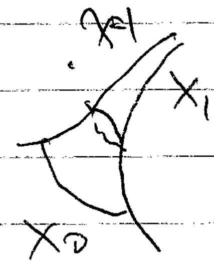
Scattering Solutions:

$(\Delta_g - \frac{1}{4} - \lambda^2)\Phi_j(x, \lambda) = 0$

$$P\Phi_1 = \begin{pmatrix} e^{-i\lambda r} + S_{11}(\lambda) e^{i\lambda r} \\ e^{-i\lambda r} + S_{21}(\lambda) e^{i\lambda r} \end{pmatrix} \qquad r > 0$$

$$P\Phi_2 = \begin{pmatrix} S_{12}\, e^{i\lambda r} \\ e^{-i\lambda r} + S_{22}(\lambda) e^{i\lambda r} \end{pmatrix}$$

<u>Construct Φ_1</u>



$\chi \in C^\infty(X)$

$\chi = 0$ near X_0

$$\Phi_1 = \chi(r)\, e^{-i\lambda r} \in C^\infty(X)$$

$$(\Delta_g - \tfrac{1}{4} - \lambda^2)\,\Phi_1 = [\Delta_g, \chi]\, e^{-i\lambda r} \in C_c^\infty$$

$$R(\lambda) = (\Delta_g - \tfrac{1}{4} - \lambda^2)^{-1} : L^2 \to L^2, \quad \operatorname{Im}\lambda > 0$$

$$R(\lambda): C_c^\infty(X) \to C^\infty(X), \quad \lambda \in \mathbb{C} \quad \text{mero.}$$

$$\Phi_1 = -R(\lambda)[\Delta_g, \chi]\, e^{-i\lambda r} + \Phi_1$$

<u>Theorem:</u> $R(\lambda): C_c^\infty(X) \to C^\infty(X)$ is meromorphic in $\mathbb{C}$ with poles of finite rank.

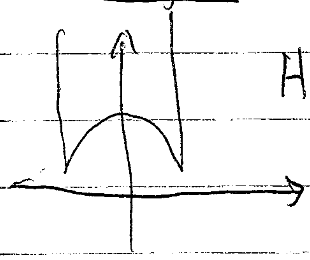
<u>Proof:</u> Similar to ~~the~~ proof of meromorphic continuation of $(-\partial_r^2 + V(r) - \lambda^2)^{-1}$ on $\mathbb{R}$ & $\operatorname{Res}_{\lambda=\lambda_0} R(\lambda) P = 0$ for $\lambda_0 \in \mathbb{R}$

$\operatorname{Res}_{\lambda=\lambda_0} R(\lambda)P = 0 \Rightarrow E_j(x,\lambda)$ is holo near $\mathbb{R}$

$$S(\lambda) = \begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) \end{pmatrix} \quad \text{unitary for } \lambda \in \mathbb{R}$$

N Cusps $\rightsquigarrow N \times N$

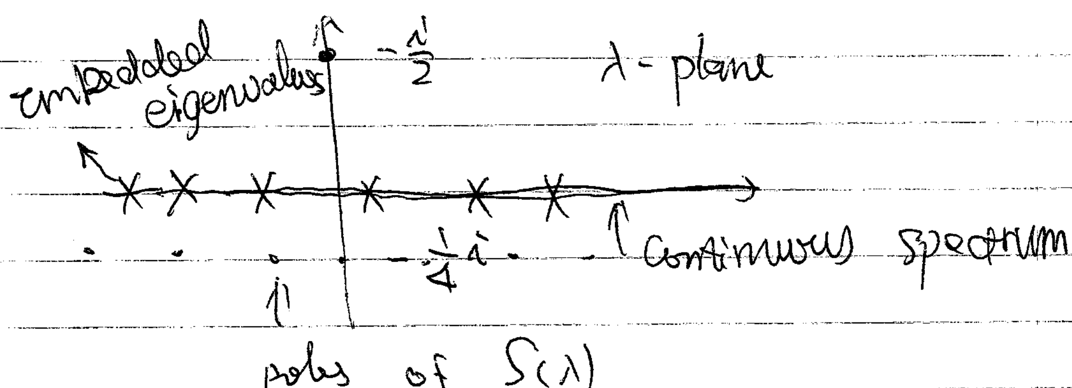
<u>Example:</u> $X = SL_2(\mathbb{Z}) \backslash \mathbb{H}^2$



$$\mathbb{H} = \left(\{ y > 0 \}, \frac{dy^2 + dx^2}{y^2} \right)$$

one cusp

$$S(\lambda) = \pi^{\frac{1}{2}} \frac{\Gamma(-i\lambda)\,\zeta(-2i\lambda)}{\Gamma(\frac{1}{2}-i\lambda)\,\zeta(1-2i\lambda)}$$



Theorem: (Colin & Verdiere 1982)
For a generic ~~metric~~ metric on X_0, there are no embedded eigenvalues.

Count!

$N(r) = \# \{ \text{poles of } R(\lambda),\ |\lambda| \leq r \}$
$N_{pp}(r) = \# \{ \text{embedded ev. } |\lambda| \leq r \}$
$\det S(\lambda) = e^{2\pi i \sigma(\lambda)}$
$N_c(r) = \sigma(r) - \sigma(-r)$

Theorem (Selberg, Müller, Parnovski)

$$N_{pp}(r) + N_c(r) = \frac{\mathrm{Vol}(X)}{2\pi} r^2 - \frac{2N}{\pi} r \log r + E(r)$$

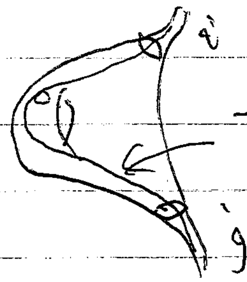
N = # of cusps

$$E(r) = \begin{cases} O(r) & \text{in general} \\ \frac{2N}{\pi}(1 - \log 2) r + o(r) & \text{under the Duistermaat–Guillemin Cond.} \end{cases}$$

Theorem (Selberg, Müller, Vodev)

$$N(r) = N_c(r) + N_{pp}(r) + o(r^2)$$

$$\forall \varepsilon > 0, \quad \# \{ \text{poles of } R(\lambda),\ \mathrm{Im}\,\lambda < -\varepsilon \} = o(r^2)$$



"Theorem": $\text{sing supp}\, \hat{S}_{ij} \subset \{$ Sojourn times between j & ith tra $\}$

Guillemin 1972, ok. for constant neg. curvature
prob. ok for neg curvature
False (?) in general.

Open Problem: (2001 Res. Cal.

Suppose X is negatively curved.
· generically only scattering poles & simple. $\Delta_g u_j = (\lambda_j^2 + \frac{1}{4}) u_j$

$$P u_j = \begin{pmatrix} a_1 e^{i\lambda_j r} \\ \vdots \end{pmatrix}$$

Show that

$$\chi \in C_c^\infty \quad \int A(\chi u_{j_k})\, \chi u_{j_k} \longrightarrow \int_{S \times X} \sigma(A)\, \pi^* \chi$$

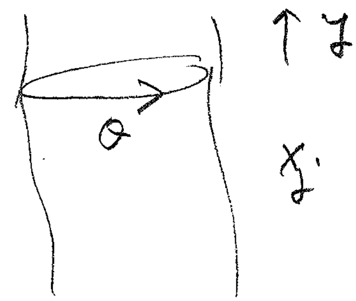
<u>Lecture 3</u>: spectral theory & scattering on surfaces with cusps.

X = 2-manifold, metric g

$X = X_0 \cup X_1 \cup X_2$ [could be $\ldots X_1 \cup \ldots \cup X_N$]

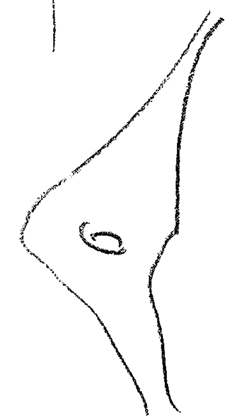
X_0 compact

$X_j = [1,\infty)_y \times S^1_\theta \qquad g|_{X_j} = \dfrac{dy^2 + a_j^{-2}\, d\theta^2}{y^2}$



$\Delta_g|_{X_j} = -y^2(\partial_y^2 + a_j^2 \partial_\theta^2)$

$d\mathrm{vol}_g|_{X_j} = \dfrac{d\theta\, dy}{a_j^2 y^2}$



$$P : L^2 \ni u \longmapsto \left(\int u|_{X_1}(y,\theta)\, d\theta,\ \int u|_{X_2}(y,\theta)\, d\theta \right) \in L^2([1,\infty)) \oplus L^2([1,\infty))$$

$\frac{dy}{y^2}$

$\mathcal{H}_{in} = \ker P \subset L^2(X)$

$$\boxed{L^2(X, d\mathrm{vol}_g) \simeq \mathcal{H}_{in} \oplus L^2([1,\infty))^2} \quad \frac{dy}{y^2}$$

$u \in C_0^\infty(X_1 \cup X_2)$

$$P \Delta_g u = \begin{pmatrix} -y^2\partial_y^2 & \\ & -y^2\partial_y^2 \end{pmatrix} P u \quad \text{on} \quad L^2((1,\infty)$$

(2) $y = e^{r}$, $\partial_y = e^{-r}\partial_r$ (L3)

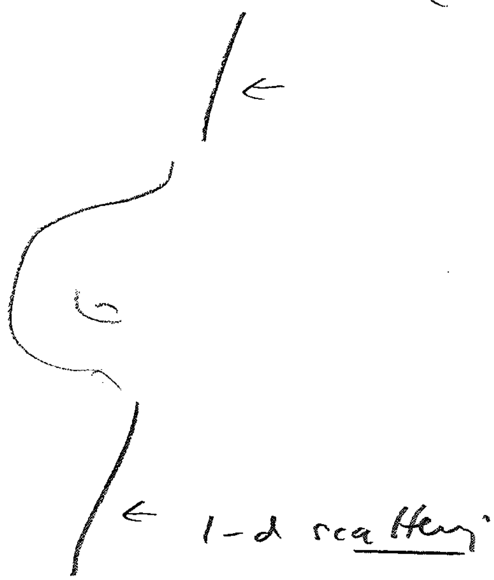
$e^{2r}(e^{-r}\partial_r)(e^{-r}\partial_r)$ on $L^2([0,\infty), e^{-r}dr)$

$e^{\frac{r}{2}}u \in L^2(e^{-r}dr) \iff u \in L^2(dr)$

$-\partial_r^2 + \partial_r$

Conjugate $e^{-\frac{r}{2}}(-\partial_r^2+\partial_r)e^{+\frac{r}{2}} = -\partial_r^2 + \frac{1}{4}$ on $L^2(dr)$

$$P\Delta_g \simeq \begin{pmatrix} -\partial_r^2+\frac{1}{4} & 0 \\ 0 & -\partial_r^2+\frac{1}{4} \end{pmatrix} P \quad \text{on } L^2(dr)^2$$



IMPORTANT FACT

$(\Delta_g - i)^{-1}(I-P)$ compact

$$\Delta_g|_{X_2} = -y^2(\partial_y^2 + a_1^2\partial_\theta^2)$$

$$\simeq \bigoplus_{n\in\mathbb{Z}} (-y^2\partial_y^2 + y^2n^2a_1^2) \quad L^2\left(\frac{dy}{y^2}\right)$$

$$\simeq \bigoplus_{n\in\mathbb{Z}} \left(-\partial_r^2 + \frac{1}{4} + e^{2r}n^2a_1^2\right) \quad L^2(dr)$$

Now $n \neq 0 \implies$

$\left(-\partial_r^2 + \frac{1}{4} + e^{2r}n^2a_1^2 - i\right)^{-1}$ is compact!

ALL scattering occurs in $\boxed{PL^2}$

3. L3

Scattering solutions (cf. TJC's lecture)

$$(\Delta_g - \tfrac{1}{4} - \lambda^2)\,\Phi_j(x,\lambda) = 0$$

$$P\,\Phi_1 = \begin{pmatrix} e^{-i\lambda r} + S_{11}(\lambda)\,e^{i\lambda r} \\ S_{21}\,e^{i\lambda r} \end{pmatrix}$$

$$P\,\Phi_2 = \begin{pmatrix} S_{12}\,e^{i\lambda r} \\ e^{-i\lambda r} + S_{22}\,e^{i\lambda r} \end{pmatrix}$$

Construct Φ_1

$\chi \in C^\infty$
supp $\chi \in X_1^\circ$
$\chi = 1$ for
$r \geq 1$

$$\Psi_1 = \chi_1(r)\,e^{-i\lambda r} \in C^\infty(X)$$

$$(\Delta_g - \tfrac{1}{4} - \lambda^2)\,\Psi_1 = [\Delta_g, \chi]\,e^{-i\lambda r} \in C_c^\infty(X)$$

Suppose [as in 1D scattering]

$$R(\lambda) = (\Delta_g - \tfrac{1}{4} - \lambda^2)^{-1} : C_0^\infty(X) \to C^\infty(X)$$

is MEROMORPHIC THEN & bdd on L^2 for $\operatorname{Im}\lambda > 0$

THEN

$$\Phi_1 = \cancel{}\,\Psi_1 - R(\lambda)\,[\Delta_g, \chi]\,e^{-i\lambda r}$$

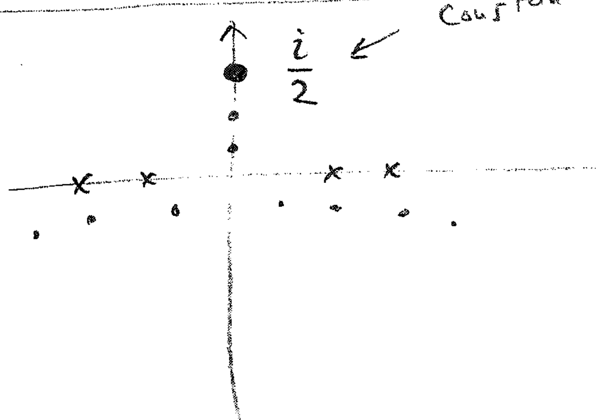
Comment: Note that $\chi e^{-i\lambda r} \notin L^2$ for $\operatorname{Im}\lambda > 0$ hence $\Phi_1 \neq 0$!

Theorem $R(\lambda): C_0^\infty(X) \to C^\infty(X)$ is meromorphic for $\lambda \in \mathbb{C}$. The poles of $R(\lambda)$, $\lambda \in \mathbb{R}$ correspond to eigenvalues and

$$\operatorname*{Res}_{\lambda=\lambda_0 \in \mathbb{R}} R(\lambda) P = 0$$

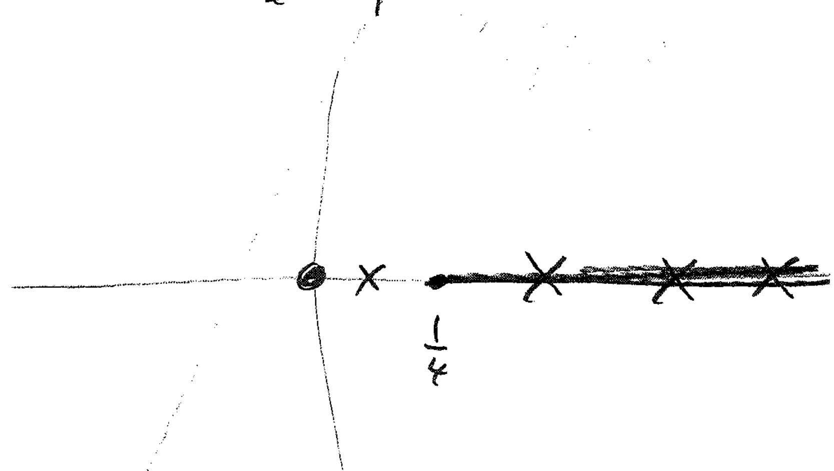
Proof is SIMILAR to the proof of meromorphy of $(-\partial_x^2 + V - \lambda^2)^{-1}$ from TJC's lecture: analytic Fredholm theory.

$z = \lambda^2 + \frac{1}{4}$



λ - plane

z - plane



Why $\operatorname*{Res}_{\lambda=\lambda_0 \in \mathbb{R}} R(\lambda) P = 0$

$(\Delta_g - \lambda^2 - \frac{1}{4})u = 0$

$\Downarrow$

$\lambda = \lambda_0$ real

$\lim_{R\to\infty} \int_{r<R} (\Delta_g)u \cdot \bar{u} - (\Delta_g)\bar{u}\, u$

$= -2i\lambda(|A|^2 - |B|^2)$

hence $|A| = |B|$

$Pu = e^{i\lambda r}A + e^{-i\lambda r}B$

Pseudo Laplacian $\Delta^\#$

$H^1 \supset H^1_\# = \{ Pu(r) = 0,\ r \geq 1 \}$

$\mathcal{D}(\Delta_\#) = H^2 \cap H^1_\#$ $\boxed{(\Delta_\# - i)^{-1} \text{ cpt}}$

allows a construction of Ks:

$Q(z)\chi = (\Delta_g - \lambda^2 - \frac{1}{4})^{-1}\chi(I + \underset{\text{cpt}}{K(\lambda)}\chi)$

Last observation $\Rightarrow$ $\Phi_j(x,\lambda)$ holomorphic in λ on $\mathbb{R}$

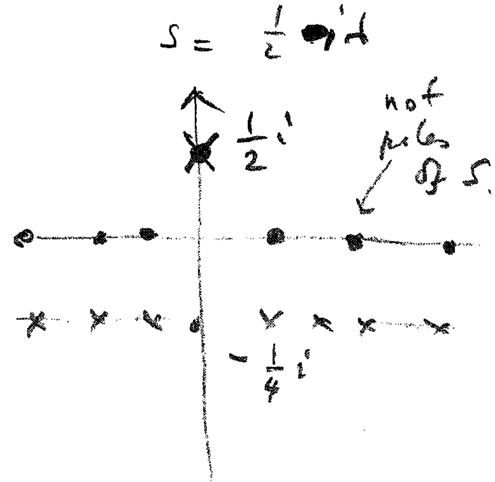
$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$ Scattering matrix

Unitary : In general $N = \#$ of cusps, $N \times N$ matrix

Example $X = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO_2(\mathbb{R})$

$N = 1$

$$\boxed{S(\lambda) = \pi^{\frac{1}{2}} \frac{\Gamma(-i\lambda)}{\Gamma(\frac{1}{2} - i\lambda)} \frac{\zeta(-2i\lambda)}{\zeta(1-2i\lambda)}}$$



~~$S(\lambda)$ unitary~~ for $\lambda \in \mathbb{R}$, meromorphic in $\mathbb{C}$

$$\boxed{S(\lambda)^* = S(\bar{\lambda})^{-1}, \quad S(\lambda)^{-1} = S(-\lambda)}$$

Objects of study

- "negative" eigenvalues $\lambda \in i\mathbb{R}_+$ poles of $R(\lambda)$
- embedded eigenvalues $\lambda \in \mathbb{R}$
- resonances of scattering $\operatorname{Im} \lambda < 0$ poles of $R(\lambda)$ & $S(\lambda)$
- scattering phase $\boxed{\sigma(\lambda) = \frac{1}{2\pi} \log \det S(\lambda)}$

· **Theorem** (Colin de Verdière)

For a generic X_0 there are no embedded eigenvalues.

$\boxed{\text{ROUGHLY: embedded ev. are eg.}^{\text{v}} \text{ of ALL } \Delta^\# \to \text{but they will move generically!}}$

Comment for $X = \Gamma \backslash \mathbb{H}^2$, Γ arithmetic: lots of ev's as $S(\lambda)$ expressed by L-f^{ns} ← Selberg, Phillips–Sarnak.

Counting:

$$N(r) = \#\{\text{poles of } R(\lambda),\ |\lambda| \le r\}$$

$$N_{pp}(r) = \#\{\text{embedded ev},\ |\lambda| \le r\}$$

$$N_c(r) = \sigma(r) - \sigma(-r) = \int_{-r}^{r} \sigma'(\lambda)\, d\lambda$$

Theorem (Selberg, Müller, Parnovski) # of cusps

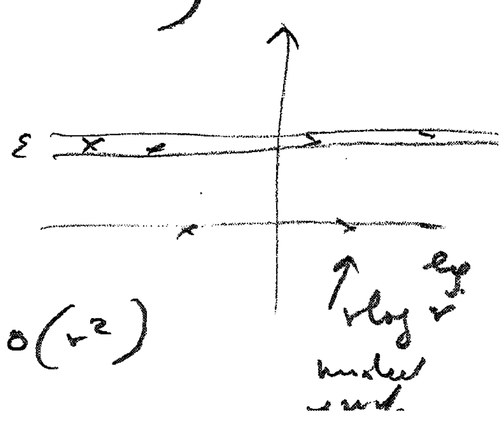
$$N_{pp}(r) + N_c(r) = \frac{\mathrm{vol}(X)}{2\pi} r^2 - \frac{2N}{\pi} r \log r + E(r)$$

$$E(r) = \begin{cases} \mathcal{O}(r) & \text{in general (Levitan–Avakumović–Hörmander)} \\ \frac{2N}{\pi}(1 - \log 2) r + o(r) & \text{(Duistermaat–Guillemin hypothesis)} \end{cases}$$

Theorem (Selberg, Müller)

$$N_c(r) = N(\varepsilon) + o(r^2)$$

$$\forall \varepsilon \quad \#\{\text{poles of } R(\lambda),\ \mathrm{Im}\,\lambda < -\varepsilon\} = o(r^2)$$



7 TRACE FORMULÆ

$$u(t) = \widetilde{\mathrm{tr}} \cos t\sqrt{\Delta_g - \tfrac{1}{4}}$$

$$\widetilde{\mathrm{tr}}\, K = \mathrm{tr}\,(I-P)K + \boxed{\text{b-tr}\, PK}$$

finite part regularization of the "noncompact part"
— see TJC's lecture later today!

<u>Theorem</u> (Poisson formula: Selberg, Müller, Guillopé–Z)

$$u(t) = \text{p.v.} \sum_{\lambda \in \mathbb{C}} e^{-i\lambda|t|}\, m_R(\lambda) - \log q\, \delta_0(t)$$

mult. of a res.

appears in the scatt. matrix

Duistermaat–Guillemin trace formula still applies here & $\boxed{\text{e.g.}}$ $u \in C^\infty(\mathbb{R}\setminus\mathcal{L})$

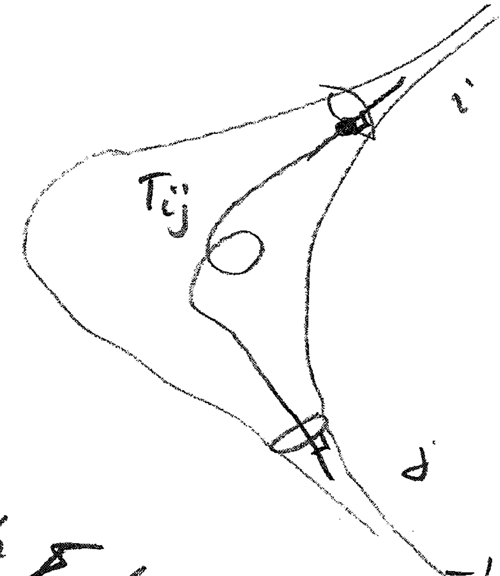
<u>Theorem</u> (Birman–Kreĭn formula; Selberg, Müller)

$$u(t) = \frac{1}{2}\,\widehat{\frac{d\sigma}{d\lambda}}(t) + \sum_{\mathrm{Im}\,\lambda \geq 0} m_R(\lambda) \cos t\lambda$$

(8.)

OPEN PROBLEM

"Theorem"



choice consistent with classif. for the scatt. matrix

Sing supp $\hat{S}_{ij} \subset \{T_{ij}\}$

$\Leftarrow$

$$\hat{S}_{ij}(t) = 2\pi \sum_{T=T_{ij}} e^{-T/2} \sum_{\pm} (t-T\pm i0)^{-\frac{1}{2}}(1+g_{ij}(A)) + u_{ij}(t)$$

u_{ij} smooth

Guillemin 1977 : OK for const neg curv. $u_{ij}(t)=0$

: OK? for neg. curv.

: FALSE? in general (bumpy metric)

OPEN PROBLEM : Shnirelman's thm for resonant states

- show resonant states are generically simple
- In the generic case show that $\forall\ \chi \in C_0^\infty(X)$

$$\int A(\chi u_{j_k})\, \chi \bar{u}_{j_k} \xrightarrow[k\to\infty]{} \int_{S^*X} \sigma(A)\, \pi^*\chi$$