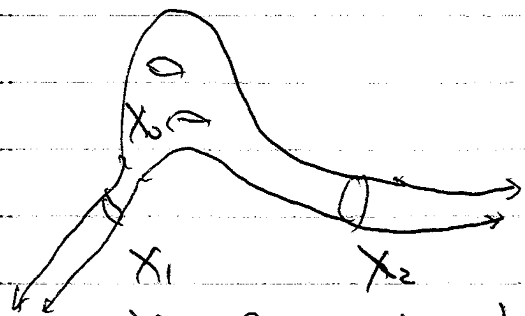
1:30 - 2:30 Sept 5 Tanya Christiansen

Manifolds with infinite cylindrical ends



(X, g) Smooth Riemannian mfd $X = X_0 \cup \bigcup_{i=1}^{N} X_i$

X_0 compact. boundary $= \bigcup_{i=1}^{N} \partial X_i$

$i = 1, \cdots, n$

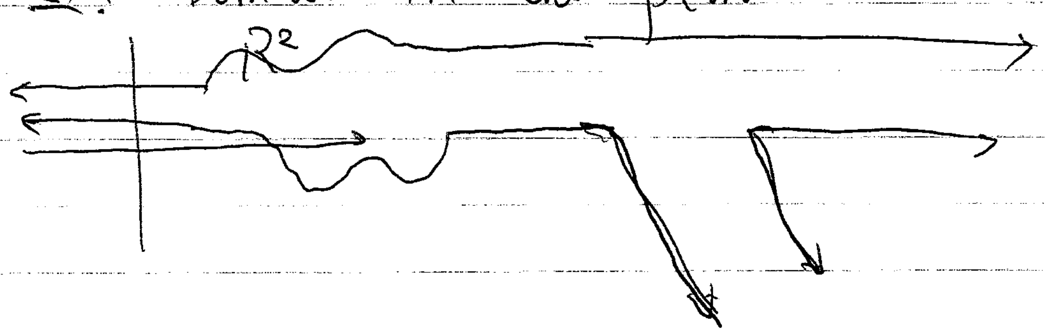
$X_i \cong [0, \infty)_s \times Y_i$

$Y_i =$ cpt mfd with no bdary

$g|_{X_i} = ds^2 + g_i$ $\quad g_i$ metric on Y_i

Generalize : Could allow boundary.

Ex. Domain in the plane.

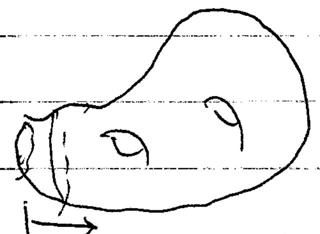


need boundary conditions. "wave guides"

② b-manifolds. (Melrose)

X_b compact smooth mfd with boundary ∂X_b

Metric h on $\mathring{X}_b$ with specific behavior near the boundary ∂X_b



x: boundary defining function. $x \in C^\infty(X_b)$ $x \geq 0$

$\{x=0\} = \partial X_b$ $\quad dx|_{\partial X_b} \neq 0$.

$$h = \left(\frac{dx}{x}\right)^2 + \tilde{h} \quad \text{with } \tilde{h} \in C^\infty(T^*X_b \otimes T^*X_b)$$

$\tilde{h}|_{\partial X_b}$ metric on the boundary

If $\exists$ a nbhd U of ∂X_b on which h is a product metric. i.e. $\tilde{h}$ is independent of x. then $\mathring{X}_b$ is isometric to a manifold with infinite cylindrical ends)

Set $x = e^{-s}$ $\quad \left|\frac{dx}{x}\right|^2 = \left(\frac{e^{-s}ds}{e^{-s}}\right)^2 = ds^2$

X manifold with infinite cylindrical ends.

what does the spectrum of Δ_X look like?

end $X_i = [a_i, \infty) \times Y_i$ $\quad (Y_i, g_i)$

Suppose $\Delta_{Y_i} \phi = \sigma^2 \phi$

~~Look at~~

$f \in C_0^\infty(X_i)$. then $\Delta_X f = \left(-\frac{d^2}{ds^2} + \Delta_{Y_i}\right) f$

Notice $\left(-\frac{d^2}{ds^2} + \Delta_{Y_i} - \lambda^2\right) \phi\, e^{\pm i\sqrt{\lambda^2-\sigma^2}\, s}\, \phi$

$= 0$

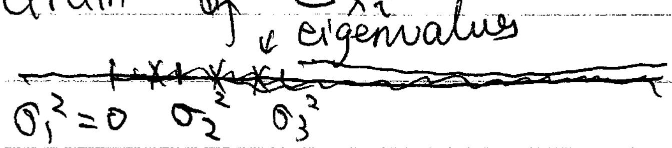
$\Rightarrow$ Continuous spectrum ~~on~~ $[\sigma^2, \infty)$

Let $Y = \coprod_{i=1}^{N} Y_i$ for $f \in C^\infty(Y)$

$(\Delta_Y f)\big|_{Y_i} = (\Delta_{Y_i} f)\big|_{Y_i}$

Let $\sigma_1^2 \le \sigma_2^2 \le \sigma_3^2 \le \cdots$ be eigenvalues of Δ_Y, repeated with multiplicity

Spectrum of Δ_{X_i}



for $\lambda \in \mathbb{R}$, multiplicity of λ^2 in cont. spectrum $= \#\{\sigma_j^2 \le \lambda^2\}$

Guillopé (infinite cylindrical ends) Melrose (b-mfds)

<u>Sampling of eigenvalue results.</u>

- Q: Find conditions that guarantee the existence of at least one eigenvalues

Eg: Exner + coauthors, Bulla + coauthors
Davies – Parnovski

Ex. Duclos-Exner. thin strip of constant width in $\mathbb{R}^2$, straight near infinity.
If non straight, Dirichlet boundary conditions, get at least 1 eigenvalue below the continuous spectrum.

- Set $N_X(\lambda)$ = number of eigenvalues of Δ_X $\leq \lambda^2$

cylindrical end: $N_X(\lambda) = O(\lambda^n)$ as $n \to \infty$
$n = \dim X$. (Parnovski, Christiansen – Zworski, Donnelly)

$\exists$ Examples X, $N_X(\lambda) \geq C_X \lambda^n$ when λ big, $C_X > 0$

Q: Generically, are there no embedded eigenvalues

<u>Scattering Theory</u>:

Let ϕ_j be an orthonormal set of eigenfunctions of Δ_Y, $\Delta_Y \phi_j = \sigma_j^2 \phi_j$.

Then there is (for $\lambda^2 > \sigma_j^2$) a Φ_j s.t.

$$(\Delta_X - \lambda^2)\Phi_j(p, \lambda) = 0$$

<u>And</u>:

$$\Phi_j\big|_{\text{ends}}(s, y, \lambda) = \frac{e^{-i r_j(\lambda) s}\phi_j(y)}{(r_j(\lambda))^{\frac{1}{2}}} + \sum e^{i r_k(\lambda) s} S_{kj}(\lambda) \frac{\phi_k(y)}{(r_k(\lambda))^{\frac{1}{2}}}$$

$r_j(\lambda) = (\lambda^2 - \sigma_j^2)$ $\quad \operatorname{Im} r_j(\lambda) > 0$

Scattering matrix for real value λ is a $N_Y(\lambda) \times N_Y(\lambda)$ matrix ($N_Y(\lambda)$ count for eigenvalues on Y)

$$S(\lambda) = \left(S_{kj}(\lambda) \right)_{k,j \le N_Y(\lambda)}$$

Facts: The scattering matrix is unitary for real λ

"The" scattering matrix depends on the choice of coordinate s.

Berman-Krein type trace Formula

$$b\text{-tr}\left[\cos\left(t\sqrt{\Delta}\right)\right] \in \mathcal{D}(\mathbb{R})$$

$$= \sum_{\mu_j \in \sigma_p(\Delta_X)} \cos\left(t\sqrt{\mu_j}\right)$$

$$+ \frac{1}{2\pi i} \int_0^a \cos \lambda t \, \frac{d}{d\lambda} \arg\left(\det S_\lambda\right) d\lambda.$$

$$+ \frac{1}{4} \operatorname{Tr} S(0) + \frac{1}{8} \sum_{\sigma_j \neq 0} \cos(\sigma_j t)$$

$$b\text{-tr}\left(\cos t\sqrt{\Delta_X}\right) = \text{Finite part}_{R \to \infty} \left(\operatorname{Tr.} \chi_{\{x: s \le R\}} \cos t\sqrt{\Delta_X} \, \chi_{\{x = s \le R\}} \right)$$

$\sigma_j > 0$

$$\lim_{\lambda \downarrow \sigma_j} \arg \det(S(\lambda)) - \lim_{\lambda \uparrow \sigma_j} \arg \det(S_\lambda) = \pi \dim\{u \in (\Delta_X - \sigma_j^2) u = 0\}$$

u is not ~~bdded~~ bounded but $\frac{1}{1+|s|} u$ is!

True for b-mfds.

Weyl Asymptotics: for cylindrical ends

$$N_X(\lambda) = \frac{1}{2\pi} \arg\det S(\lambda) = C_n \, b\text{-vol}(X) \, \lambda^n + O(\lambda^{n-1})$$

(Parnovski, Christiansen-Zworski)

<u>Prize</u>: Is this true for b-mfds?