An Introduction to L^2 Cohomology

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What is L^2 cohomology?

The de Rham complex is the cochain complex of (smooth) exterior differential forms on some smooth manifold M, with the exterior derivative as differential.

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de Rham Theorem: The de Rham cohomology $H^k_{\mathrm{dR}}(M) \stackrel{\mathrm{def}}{=} \ker d_k / \mathrm{Im}\ d_{k-1}$ is isomorphic to the singular cohomology:

$$H^k_{\mathrm{dR}}(M) \cong H^k(M; \mathbf{R}).$$

Introducing Geometry: Let g be a Riemannian metric on M. Then g induces an L^2-metric on $\Omega^k(M)$. As usual, let δ denote the formal adjoint of d. In terms of a choice of local orientation for M, we have $\delta = \pm * d *$, where $*$ is the Hodge star operator. Define the Hodge Laplacian to be

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$$\text{Topology} \overset{\text{Geometry}}{\longleftrightarrow} \text{Analysis}$$

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Restricting to square integrable differential forms $\longrightarrow$ L^2 cohomology

More precisely, let (Y, g) denote an open (possibly incomplete) Riemannian manifold.

$\Omega^i = \Omega^i(Y)$ the space of C^∞ i-forms on Y and $L^2 = L^2(Y)$ the L^2 completion of Ω^i with respect to the L^2-metric. Define d to be the exterior differential with the domain

$$\operatorname{dom} d = \{\alpha \in \Omega^i(Y) \cap L^2(Y);\ d\alpha \in L^2(Y)\}.$$

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The L^2 cohomology depends only on the quasi-isometry class of the metric.

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- Finite cone $C(N) = C_{[0,1]}(N) = (0,\ 1) \times N$, N a closed manifold of dimension n, with the conical metric $dr^2 + r^2 g_N$:

$$H^i_{(2)}(C(N)) = \begin{cases} H^i(N) & \text{if } i < (n+1)/2, \\ 0 & \text{if } i \geq (n+1)/2. \end{cases}$$

Cheeger

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Also, there is a natural map from the compact supported cohomolgy to the L^2 cohomology which is also neither injective nor surjective in general:

$$H^i_c(Y) \longrightarrow H^i_{(2)}(Y).$$

L^2 completion

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d has well defined strong closure $\bar{d}$ in L^2: $\alpha \in \mathrm{dom}\,\bar{d}$ and $\bar{d}\alpha = \eta$ if there is a sequence $\alpha_j \in \mathrm{dom}\,d$ such that $\alpha_j \to \alpha$ and $d\alpha_j \to \eta$ in L^2.

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Similarly, δ has strong closure $\bar{\delta}$.

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Then, the natural map,

$$\iota_{(2)} : \; H^i_{(2)}(Y) \longrightarrow H^i_{(2),\#}(Y) \, ,$$

is always an isomorphism.

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The reduced L^2-cohomology is defined by

$$\bar{H}^i_{(2)}(Y) = \ker \bar{d}_i / \overline{\mathrm{Im}\, \bar{d}_{i-1}}\,.$$

Now we define the space of L^2-harmonic i-forms $\mathcal{H}^i_{(2)}(Y)$ to be the space

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The most basic result here is the Kodaira decomposition,

$$L^2 = \mathcal{H}^i_{(2)} \oplus \overline{d\Lambda_0^{i-1}} \oplus \overline{\delta\Lambda_0^{i+1}},$$

an orthogonal decomposition which leaves invariant the subspaces of smooth forms. Here subscript "0" = having compact support.

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Surjectivity: If Im $\bar{d}$ is closed, then the Hodge map is surjective.

In particular, this holds if the L^2-cohomology is finite dimensional.

Injectivity: The issue of injectivity of the Hodge map has to do with the L^2 Stokes theorem.

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We say that Stokes' theorem holds for Y in the L^2 sense, if

$$\langle \bar{d}\alpha, \ \beta\rangle = \langle \alpha, \ \bar{\delta}\beta\rangle$$

for all $\alpha \in \mathrm{dom}\, \bar{d}$, $\beta \in \mathrm{dom}\, \bar{\delta}$;

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or equivalently, one has

$$\langle d\alpha,\ \beta\rangle = \langle \alpha,\ \delta\beta\rangle$$

$\alpha \in \operatorname{dom} d$, $\beta \in \operatorname{dom} \delta$.

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Thus, the Hodge map is injective in this case.

Moreover,

$$\mathcal{H}^i_{(2)}(Y) \cong \bar{H}^i_{(2)}(Y),$$

and

$$H^i_{(2)}(Y) = \bar{H}^i_{(2)}(Y) \oplus \overline{\mathrm{Im}\, \bar{d}_{i-1}}/\mathrm{Im}\, \bar{d}_{i-1}.$$

Here, by the closed graph theorem, the last summand is either 0 or infinite dimensional.

Summarize

If the L^2-cohomology of Y is finite dimensional and Stokes' Theorem holds on Y in the L^2-sense, then the L^2-cohomology of Y is isomorphic to the space of L^2-harmonic forms and therefore, when Y is orientable, Poincáre duality holds as well. Consequently, the L^2 signature of Y is well-defined in this case.

Summarize

If the L^2-cohomology of Y is finite dimensional and Stokes' Theorem holds on Y in the L^2-sense, then the L^2-cohomology of Y is isomorphic to the space of L^2-harmonic forms and therefore, when Y is orientable, Poincáre duality holds as well. Consequently, the L^2 signature of Y is well-defined in this case.

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Gaffney: L^2 Stokes theorem holds for a complete Riemannian manifolds.

Cheeger: L^2 Stokes theorem holds for a finite cone $C(N)$ if and only if L^2 Stokes holds for N, which is the case if N is closed.

Remarks

- coefficients
- Dolbeault
- topological interpretation

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Let

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denote a fibration of closed oriented smooth manifolds.

Denote by $C_\pi M$ the mapping cylinder of π. This is obtained by attaching a cone to each of the fibres. Indeed, we have

$$C_{[0,1]}(Z) \to C_\pi M \to B\,.$$

The space $C_\pi M$ also comes with a natural quasi-isometry class of metrics. A metric can be obtained by choosing a submersion metric on M:

$$g_M = \pi^* g_B + g_Z.$$

Then, on the nonsingular part of $C_\pi M$, we take the metric,

$$g_1 = dr^2 + \pi^* g_B + r^2 g_Z.$$

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The general class of spaces with non-isolated conical singularities as above can be described as follows. A space X in the class will be of the form

$$X = X_0 \cup X_1 \cup \cdots \cup X_k,$$

where X_0 is a compact smooth manifold with boundary, and each X_i (for $i = 1, \ldots, k$) is the associated mapping cylinder, $C_{\pi_i} M_i$, for some fibration, (M_i, π_i), as above.

Generalized Thom spaces

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Question: What is the L^2 signature of T?

Example

Let $\xi \xrightarrow{\pi} B$ be a vector bundle of rank k. Then we have the associated sphere bundle:

$$S^{k-1} \longrightarrow S(\xi) \xrightarrow{\pi} B.$$

The generalized Thom space constructed out of this fibration coincides with the usual Thom space equipped with a natural metric.

In this case,

$$\mathrm{sign}_{(2)}(T) = -\mathrm{sign}(D(\xi)),$$

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Let Φ denote the Thom class and χ the Euler class. Then the Thom isomorphism gives

$$\begin{array}{ccccc}
H^*(D(\xi), S(\xi)) & \otimes & H^*(D(\xi), S(\xi)) & \to & \mathbf{R} \\
\uparrow \pi^*(\cdot) \cup \Phi & & \uparrow \pi^*(\cdot) \cup \Phi & & \\
H^*(B) & \otimes & H^*(B) & \to & \mathbf{R} \\
\phi & & \psi & \to & [\phi \cup \psi \cup \chi][B]\,.
\end{array}$$

Thus, $\mathrm{sign}_{(2)}(T)$ is the signature of this bilinear form on $H^*(B)$.

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Let (E_r, d_r) be the E_r-term with differential, d_r, of the Leray spectral sequence of the fibration in the construction of T. Define a pairing

$$\begin{array}{ccc} E_r \otimes E_r & \rightarrow & \mathbf{R} \\ \phi \otimes \psi & \mapsto & \langle \phi \cdot d_r \psi, \xi_r \rangle \end{array},$$

where ξ_r is a basis for E_r^m naturally constructed from the orientation. In case $m = 4k-1$, when restricted to $E_r^{\frac{m-1}{2}}$, this pairing becomes symmetric. We define τ_r to be the signature of this symmetric pairing and put

$$\tau = \sum_{r \geq 2} \tau_r .$$

Theorem (Cheeger-D.)

Assume that the fibre Z is either odd dimensional or its middle dimensional cohomology vanishes. Then the L^2-signature of the generalized Thom space T is equal to $-\tau$:

$$\mathrm{sign}_{(2)}(T) = -\tau.$$

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Corollary

For a space X with non-isolated conical singularity, the L^2-signature is given by

$$\mathrm{sign}_{(2)}(X) = \mathrm{sign}(X_0) + \sum_{i=1}^{k} \tau(X_i)\,.$$

Corollary

Assume that the fibre Z is either odd dimensional or its middle dimensional cohomology vanishes. Then we have the following adiabatic limit formula for the eta invariant of the signature operator.

$$\lim_{\epsilon\to 0}\eta(A_{M,\epsilon}) = \int_B \mathcal{L}(\frac{R^B}{2\pi})\wedge\tilde{\eta}+\tau.$$

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This is a special case of a general result of **D.**.