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L^2 cohomology of QALE Spaces

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Quasi Asymptotically Locally Euclidean (QALE) spaces:

were introduced by D. Joyce who constructed new examples of complete non-compact Riemannian manifolds with special holonomy, eg Kähler-Einstein, hyperkähler & G_2 type holonomy.

I) The basic building block for QALE spaces are ALE (asymp. loc Eucl) spaces.

a) Ex The singular space $\mathbb{C}^2/\pm I$ has an isolated singularity at 0. This can be resolved to form the space $T^*\mathbb{P}^1(\mathbb{C})$. We see this as follows: $\mathbb{P}^1(\mathbb{C}) = \{L \subset \mathbb{C}^2 \mid L = a \text{ cx line}\}$.

$$T\mathbb{P}^1(\mathbb{C}) = \{(L, \alpha) \mid \alpha : L \to \mathbb{C}^2/L\}$$

$$T^*\mathbb{P}^1(\mathbb{C}) = \{(L, \xi) \mid L \text{ is a line and } \xi : \mathbb{C}^2/L \to L\}$$

So we can consider the subspace

$$T^*\mathbb{P}^1(\mathbb{C}) \setminus Z \cong \{\xi : \mathbb{C}^2 \to \mathbb{C}^2 \mid \xi \circ \xi = 0,\ \xi \neq 0\}$$

Then

$$T^*\mathbb{P}^1 - Z = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \;\middle|\; x^2 + yz = 0,\ (x,y,z) \neq (0,0,0) \right\}$$

Now identify the regular part

$$\mathbb{C}^2 \setminus \{0\}/\pm I \longrightarrow \begin{pmatrix} ab & a^2 \\ -b^2 & -ab \end{pmatrix} \in T^*\mathbb{P}^1 - Z.$$

Let $r = \sqrt{|a|^2 + |b|^2}$ on $\mathbb{C}^2 - \{0\}/\pm I$, $f(a,b) = \sqrt{1 + r^4} + 2\ln r - \ln(\sqrt{1+r^4} + 1)$.

Then $\omega = i\partial\bar\partial f$ is a Kähler form which extends to $T^*\mathbb{P}^1(\mathbb{C})$.
The associated Riemannian metric g is called

the Eguchi - Hansen metric. (E-H)

Properties of E-H metric; • Kähler - Einstein

• scalar curvature $= 0$

• on $\mathbb{C}^2 - \{0\}/\pm I$, $g_{EH} = $ euclidean $+ O(\frac{1}{r^4})$.

b) Definition of ALE A Riemannian mfld (M^{2n}, g) is called ALE asymptotic to $\mathbb{C}^n/\Gamma$, where $\Gamma \subset U(n)$ is finite without fixed pt on $\mathbb{C}^n \setminus \{0\}$, if there is a cpt $K \subset\subset M$ and a diffeo. $\varphi: \mathbb{C}^n \setminus B^n/\Gamma \to M \setminus K$ such that $\varphi^* g = \text{eucl} + o(1)$ (order zero)

II) QALE spaces

a) Example $\Gamma = \mathbb{Z}_4 \subset SU(3)$ generated by $\alpha(z_1, z_2, z_3) = (-z_1, iz_2, iz_3)$. Then α has no fixed pt except $(0,0,0)$, but α^2 does: $\alpha^2(z_1, z_2, z_3) = (z_1, -z_2, -z_3)$.

Then if we set $\Lambda = \{Id, \alpha^2\}$ we get $\mathbb{C}^3/\Lambda \cong \mathbb{C} \times \mathbb{C}^2/\{\pm I\}$ which can be resolved to $\mathbb{C} \times T^*\mathbb{P}^1(\mathbb{C})$. Now α acts on $\mathbb{C} \times T^*\mathbb{P}^1(\mathbb{C})$ by $\bar{\alpha}$, $\bar{\alpha}^2 = Id$, with fixed pt set $\{0\} \times \mathbb{P}^1$.

We can again resolve

$$\mathbb{C} \times T^*\mathbb{P}^1(\mathbb{C}) / \langle 1, \bar{\alpha} \rangle \quad \text{to a manifold } Y$$

Such that there is a kähler Einstein metric g on Y which on $\mathbb{C} \times T^*\mathbb{P}^1(\mathbb{C}) - K$ is $g = \text{eucl}_{\mathbb{C}} + g_{EH} + o($

b) <u>Definition of QALE:</u> By induction. Will only look at the first few steps. So consider <u>QALE of depth 2</u>: Let $\Gamma \subset U(n)$ be a finite group such that the set

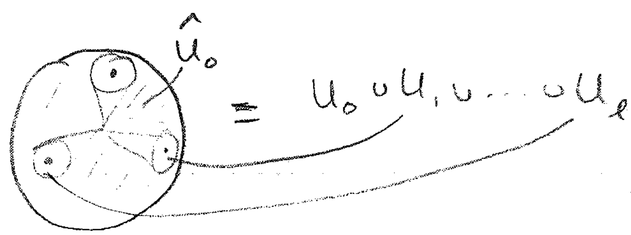
$$\{v \in \mathbb{C}^n \mid \exists \gamma \in \Gamma \text{ with } \gamma v = v\} = H_1 \cup \cdots \cup H_k$$

where $H_i \cap H_j = \{0\}$ if $i \neq j$.

(ex: $\Gamma \subset SU(3)$ finite, or $\Gamma \subset Sp(2)$ finite)

Then the geometry of $\mathbb{C}^n - \{0\}/_\Gamma = C(S^{2n-1})/_\Gamma - \{\text{cone pt}\}$

picture



where U_0 is a cone over $(S^{2n-1} - (\cup H_i)^\varepsilon)/_\Gamma$

(remove small cone around singular pts on sphere)

and each U_i (up to a finite cover) is a subset of

$$H_i^\perp - \{0\}/_{A_i} \oplus H_i \qquad \text{where } A_i = \{\gamma \in \Gamma \mid \gamma|_{H_i} = Id_{H_i}\}.$$

The A_i act without fixed pt on $H_i^\perp - \{0\}$ by assumption, so the quotient in the first summand can be resolved to an ALE space.

U_i is the subset

$$U_i \simeq \{(y, z) \in H_i^\perp - \{0\}/_{A_i} \oplus H_i \mid |y| \leq \varepsilon |z|\}$$

So it can also be resolved.

So define a <u>QALE space asymptotic to $\mathbb{C}^n/_\Gamma$</u> as an Riemannian mfld (M^{2n}, g) such that outside a cpt $K \subset\subset M$,

$(M \setminus K) = \hat{U}_0 \cup \hat{U}_1 \cup \cdots \cup \hat{U}_\ell$

where $\hat{U}_0$ is diffeomorphic to $\{x \in U_0, |x| \geq R\}$ and g is asymptotic to Euclidean. For some ALE space Y_i asymptotic to $H_i^\perp / A_i$, then $\hat{U}_i$ is diffeomorphic to $\{(y,z) \in Y_i \times H_i \mid d_{Y_i}(y,0) \leq \varepsilon |z| \mid \leq |z|\}$ and g is asymptotic to $g_{Y_i} \oplus$ Eucl.

III L^2 Cohomology

Our goal is to calculate the L^2 cohomology for such spaces.

Def If (V,g) is a Riemannian manifold then the <u>L^2 cohomology of V</u> is

$$H_2^k(V) \simeq Z_2^k(V) / B_2^k(V)$$

where
$$Z_2^k(V) = \{\alpha \in L^2(\Lambda^k T^*V) \mid d\alpha = 0 \text{ in the weak sense}\}$$
$$B_2^k(V) = \{d\beta \mid \beta \in L^2(\Lambda^{k-1} T^*V), d\beta \in L^2 \text{ in the weak sense}\}$$

The <u>reduced L^2 cohomology of V</u> is

$$\bar{H}_2^k(V) \simeq Z_2^k(V) / \overline{B_2^k(V)}.$$

These spaces depend only on the quasiisometry class of g, so we may make some assumptions about the structure of g.

We also have for g complete that

$$\overline{H}_2^k(V) \simeq \{\alpha \in L^2(\Lambda^k T^*V) \mid d\alpha = d^*\alpha = 0 \text{ in the weak sense}\}$$
$$\simeq \{\alpha \in L^2(\Lambda^k T^*V) \mid (dd^* + d^*d)\alpha = 0 \text{ in the weak sense}\}.$$

On QALE spaces, $H_2^k(V)$ is always ∞ dim'l, so we will consider $\overline{H}_2^k(V)$.

The advantage of $H_2^k(V)$ is that one can use tools from algebraic topology, such as Mayer-Vietoris. A priori, there is no M-V sequence for $\overline{H}_2^k(V)$. Nevertheless, we will try to use this type of approach.

IV Local calculations.

We will use three results without proof:

a) Prop (C., Melrose) if (M^{2n}, g) is an ALE space asymptotic to $\mathbb{C}^n/\Gamma$ then $\overline{H}_2^k(M) \simeq \operatorname{Im}(H_c^k(M) \to H^k(M))$.

b) An infinite truncated cone: Let $C_1(N) = \{(r,\theta) \in]1,\infty[\times N\}$ with metric $dr^2 + r^2 h$ where N is compact with boundary. Then

$$\overline{H}_2^k(C_1(N)) = \begin{cases} \{0\} & k \leq \dim C_1(N)/2 \\ H^k(N) & k > \dim C_1(N)/2 \end{cases}$$

c) L^2 cohomology of $\hat{G} = \{(y,z) \in Y \times (\mathbb{C}^k - B^k) \mid d(y,o) \le \varepsilon|z|, R \le |z|\}$

for Y an ALE space, is given by

$$\bar{H}_2^p(\hat{G}) \cong \bar{H}_2^{p-(2k-1)}(Y)$$

V How can we glue these local calculations?

When we try to follow the proof of M-V, we need to understand primitives. So to understand what might work for $\bar{H}_2^k$ we need to understand good primitives of L^2 forms.

a) Finding good primitives: Let (V,g) be a Riemannian manifold and $\alpha \in L^2(\Lambda^k T^*V)$, $d\alpha = 0$ such that for a certain φ, $\alpha = d\varphi$. Under what conditions on φ can we say $[\alpha] = 0$ in $\bar{H}_2^k(V)$ (ie, in reduced L^2 coh.)? Of course if $\varphi \in L^2(\Lambda^{k-1} T^*V)$, this is true, but even if it is not, $[\alpha]$ could $= 0$ in $\bar{H}_2^k(V)$. We need to define a sequence of $\varphi_i \in \operatorname{dom} d$ such that $\alpha = L^2$ limit of $d\varphi_i$. Use cut-off fns. χ_i. Then we get

$$\alpha = \lim_{L^2} d(\chi_i \varphi)$$

$$= \chi_i d\varphi + \underbrace{d\chi_i \wedge \varphi}$$ if we can control

Precisely, we get this if ...

Lemma (P. Li, N. Hitchin, J. McNeal)
Assume α belongs to the L^2_{loc} domain of d, $r: V \to \mathbb{R}_+$ is a proper function with bounded gradient and $\rho: [1,\infty) \to [1,\infty)$ is such that

$$(*) \quad \int_1^\infty \frac{1}{\rho(r)}\, dr = \infty, \quad \text{Then if } \frac{\varphi}{\rho} \in L^2, \text{ we get } [\alpha] = 0$$
in $\bar{H}^k_2(V)$.

Define $\mathcal{E}^{k-1}_\rho = \{\varphi \in L^2_{loc} \mid d\varphi \in L^2, \frac{\varphi}{\rho} \in L^2\}$ as our new space of primitives. Then if ρ satisfies $(*)$ and $d: \mathcal{E}^{k-1}_\rho \to L^2$ has closed range then

$$\overline{B_2^k(V)} = \overline{d\mathcal{E}^{k-1}_\rho}$$

b) In this setting we have a Mayer-Vietoris exact sequence:
Let $V = \mathcal{O}_1 \cup \mathcal{O}_2$. Assume

(i) $\mathcal{E}^{k-1}_\rho(\mathcal{O}_1) \oplus \mathcal{E}^{k-1}_\rho(\mathcal{O}_2) \to \mathcal{E}^{k-1}_\rho(\mathcal{O}_1 \cap \mathcal{O}_2) \to \{0\}$ is exact

(ii) $d: \mathcal{E}^{k-1}_\rho \to L^2$ is closed on $\mathcal{O}_1 \cup \mathcal{O}_2$, $\mathcal{O}_1 \cap \mathcal{O}_2$, $\mathcal{O}_1$, $\mathcal{O}_2$

Then there is an exact MV sequence involving weighted reduced L^2 cohomology.

$$\bar{H}^{k-1}_\rho(\mathcal{O}_1) \oplus \bar{H}^{k-1}_\rho(\mathcal{O}_2) \to \bar{H}^{k-1}_\rho(\mathcal{O}_1 \cap \mathcal{O}_2) \to \bar{H}^k_2(\mathcal{O}_1 \cup \mathcal{O}_2) \to \bar{H}^k_2(\mathcal{O}_1) \oplus \bar{H}^k_2(\mathcal{O}_2)$$
$$\to \bar{H}^k_2(\mathcal{O}_1 \cap \mathcal{O}_2)$$

<u>Note</u>: we cannot always iterate this farther to the left, as ρ satisfying (*) does not imply ρ^2 does, and we'd need ρ^2 in the iteration.

c) <u>A result</u>. With this tool

<u>Thm</u> (Carron) Assume that $\Gamma \subset SU(n)$ is a finite group (of depth 2). Let $\pi : X \longrightarrow \mathbb{C}^n/\Gamma$ be a crepant resolution of $\mathbb{C}^n/\Gamma$ endowed with a QALE metric.
Then
$$\bar{H}^k_2(X) = \operatorname{Im}(H^k_c(X) \to H^k(X)).$$

For non-crepant resolutions, there is also a result, but it is more complicated.