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The signature operator on Witt spaces

8 September 2008

(Joint work with Albin, Mazzeo, Leichtman)

1. The Index Package

Consider closed compact manifolds. In this case, index theory is very complete. There is an "index package":

Let X be a smooth cpt mfld, let $\Gamma = \pi_1(X)$ or any discrete group.

Given $r: X \to B\Gamma$, (classifying space), $\tilde{X} = r^* E\Gamma$ (eg, univ cover if $\Gamma = \pi_1(X)$ and r is the universal map)

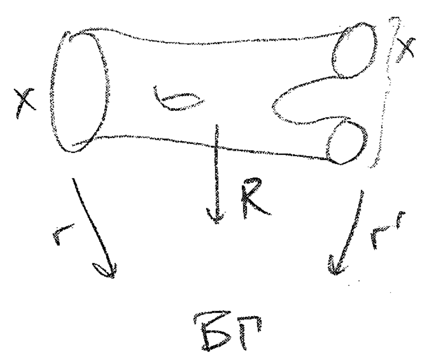
Then we get a flat vector bundle $\mathcal{V} = \tilde{X} \times_r C^*_r\Gamma$ $\quad (\mathbb{C}\Gamma)^-$ acts on $\ell^2(\Gamma)$

↑ some C^* algebra

The signature operator twisted by this bundle is $D^{\text{sign}}_{\mathcal{V}}$ and associated to this is an index class $\text{Ind}(D^{\text{sign}}_{\mathcal{V}}) \in K_*(C^*_r\Gamma)$

① Associated to $(X, r: X \to B\Gamma)$ is an index class $\text{Ind}(D^{\text{sign}}_{(X,r)}) \in K_*(C^*_r\Gamma)$

② $\text{Ind}: \Omega_*(B\Gamma) \to K_*(C^*_r\Gamma)$ ie, Ind is a cobordism invariant.



③ Can consider the chain complex in the local system:

$$\longrightarrow C^j(X, \mathcal{V}) \longrightarrow C^{j+1}(X, \mathcal{V}) \longrightarrow \cdots$$

This is an algebraic Poincaré complex, thus we can define a topological Mishchenko signature

$$\sigma(X, r) \in L^*(C^*_r\Gamma) \xrightarrow{\cong} K_*(C^*_r\Gamma)$$

and again this is a bordism invariant, ie,

$$\sigma : \Omega_*(B\Gamma) \longrightarrow L^*(C^*_r\Gamma)$$

(4) $\sigma[X,r] = \mathrm{Ind}(\mathcal{D}^{\mathrm{sign}}_{[X,-]})$

topological object — analytic object

(5) There are index formulae for $\mathrm{Ind}(\mathcal{D}^{\mathrm{sign}}_{[X,r]})$.
(A-S, plus higher Moscovici indices)

II. Manifolds with boundary: Extending this package requires the appropriate tools from microlocal analysis.
Some has been established (1, 5 Piazza, Leichtman, ?
3 Weinberger, Lück, Kreck, Lesc
4 - still open)

For the analytic part, we use the b-calculus of Melrose, (Use parametrix for D^{sign}_{ν})

III; Stratified manifolds What of 1-5 can be generalized to stratified manifolds?
This question is hard to answer in generality as the signature operator generally has many different extensions. Thus we restrict to Witt spaces where the extension is unique.

Assume $\hat{X}$ is a stratified pseudomanifold endowed with a Thom-Mather stratification, i.e.

stratified pseudomfld ⇒ $\hat{X}_{-1} \subseteq \hat{X}_0 \subseteq \cdots \subseteq \hat{X}_{m-1} \subseteq \hat{X}_m = \hat{X}$

$\hat{X}_{-1} = \phi$ and $Y_i = \hat{X}_i - \hat{X}_{i-1}$ are manifolds, and each stratum Y_i has a tubular nbhd T_i with a fibration

$$c(Z_i) \longrightarrow T_i \xrightarrow{\phi_i} Y_i$$

(if $\hat{Y}_i$ are not connected, the Z_i can be different over different components).

Thom-Mather ⇒ and $\exists\ \rho : T_i \to [0, r)$ where ρ is smooth on the regular part of T_i and $Y_i = \rho_i^{-1}(0)$.

(+ axioms about compatibility conditions)

Then the Witt condition says for odd $(2\ell+1)$ co-dimensional strata Y, the $IH^{\ell}_{\substack{\text{lower}\\ \text{middle}}}(Z) = 0$.

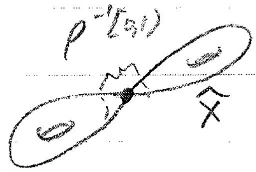
In a paper from 1980's, Cheeger considered the signature operator on such manifolds & sketches proof in general case. One motivation for the current work is to flesh this out.

GOAL: Establish versions of 1-5 for Witt spaces.

Notes: Must make sense of cobordism group in this setting. Seigal has done this, ie $\Omega_*^{\text{witt}}(B\Gamma)$ has been defined, and its coefficient has been computed.

IV Isolated singularities

Start with $\hat{X}^{2n}$ with isolated singularities, and radial variable, ρ. given by Thom-Mather structure.



Let X be the manifold with boundary
$X = \hat{X} \setminus \rho^{-1}[0,1)$, $\partial X = \{x = 0\} \cong Z$
where $x = \rho - 1$ on X.

Endow X with a conical metric $g = dx^2 + x^2 g_Z = x^2 \tilde{g}$
Then the signature operator is near ∂X given by
$$D_g^{sign} \sim \frac{\partial}{\partial x} + \frac{D_Z}{x}$$
so
$$x D_g^{sign} \sim x\frac{\partial}{\partial x} + D_Z$$ is an <u>elliptic b-operator</u>

The idea is to study D_g^{sign} using <u>b-calculus</u>.
In more general setting, use results from complete case.

<u>Remark</u>: This operator is equal to the signature operator for the conformal <u>complete</u> metric: $x D_g^{sign} = D_{\tilde{g}}^{sign}$ up to a term of order 0 that can be written explicitly.

b-elliptic operators

<u>Def</u> Let X be a manifold with boundary $\partial X = \{x = 0\}$.
A <u>b-differential operator</u> is generated by vector fields tangent to ∂X, so locally, $\{x\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}}\}$
where $x = $ bdf and y_i are local variables on ∂X.
We denote the algebra of such operators by $\mathrm{Diff}_b^*(X)$.

Locally $P \in \mathrm{Diff}_b^m(X)$ can be written as

$$P = \sum e_{\alpha,j}(x,y)\left(\frac{\partial}{\partial y}\right)^\alpha \left(x\frac{\partial}{\partial x}\right)^j.$$

P is b-elliptic if it is elliptic on the interior and near ∂X, $\sum e_{\alpha,j}(x,y)(i\xi)^\alpha(i\lambda)^j$ is invertible off $(\xi,\eta)=\vec{0}$.

To study P we must study the model operator at ∂X:

$$I(P) = \sum a_{\alpha,j}(0,y)\left(\frac{\partial}{\partial y}\right)^\alpha (s\partial_s)^j$$

and let it act on $L^2(\mathbb{R}^+_s \times \partial X_y;\, x\, dx \wedge d\mathrm{vol}_{\partial X})$.

If we take the Mellin transform, we get

$$I(P,\lambda) = \sum a_{\alpha,j}(0,y)\left(\frac{d}{dy}\right)^\alpha (i\lambda)^j$$

which is a 1-parameter family of operators on ∂M.

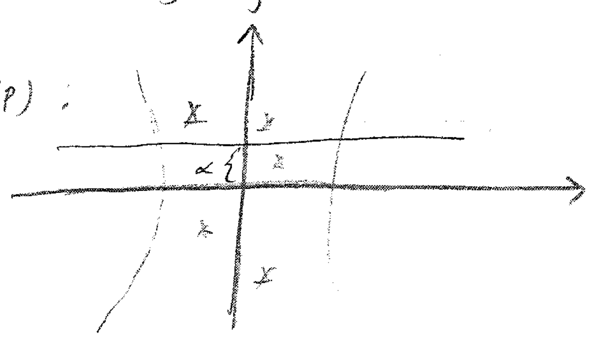
weighted Sobolev space

weighted L^2 space

<u>Thm</u> (Melrose) $P: x^a H_b^m(X) \longrightarrow x^a L_b^2$ is Fredholm (for a degree m elliptic b-operator P) iff $a \notin \mathrm{Im}(\mathrm{spec}_b(P))$

imaginary part

$\mathrm{spec}_b(P)$:



$\mathrm{Spec}_b(P) = \{z \in \mathbb{C} \mid I(P,z) \text{ is not invertible}\}$

(using basis $(dx, x\,dy)$) (using basis $(\frac{dx}{x}, dy)$

Using $L^2_g = x^{-\frac{f+1}{2}} L^2_{\tilde{g}}$, where $f = \dim \partial X$

$\frac{dx}{x}\, dy$

$\left(= x^n L^2_{\tilde{g}} \qquad x = \dim X\right)$

we see that to use this result on cones,

we are considering $x\Delta_g$ on $x^{-\frac{(f+1)}{2}} L^2_g$

which is not generally Fredholm.

Thm (Lesch, ...)

1) $\mathcal{D}_{max}(D_g) = \mathcal{D}_{min}(D_g)$ iff $\mathrm{Im}(\mathrm{spec}_b(P)) \cap \left(-\frac{(f+1)}{2}, -\frac{(f-1)}{2}\right) = \emptyset$

Cheeger → 2) D_g is self-adjoint under this condition

3) D_g is Fredholm.

Remarks

- Theorem is proved using b-calculus only
- 3 (the most delicate) employs fine properties of the parametrix acting on $x^{-\varepsilon}(x^{-\frac{(f+1)}{2}} L^2_{\tilde{g}})$

The conclusion is that Cheeger's result in the isolated sing setting can be proved using Melrose.

V. Cone bundle singularities

Now consider a space where the singularity is a bundle of cones. The Witt condition here says the links, Z_0 must either be odd dim'l or must have $H^{\frac{f}{2}}(Z_0) = \{0\}$. $f = \dim Z_0$

Then we can cut out the singularity and put a (degenerating) metric on the resulting manifold with boundary, X.

Then $Z_0 \to \partial X$, and the metric is

$$\begin{array}{l} Z_0 \to \partial X \\ \downarrow \phi_0 \\ Y_0 \end{array} \qquad g = dx^2 + x^2 g_{Z_0} + \phi_0^* g_{Y_0}$$

Consider the associated complete metric:

$$\hat{g} = \frac{dx^2 + \phi_0^* g_{Y_0}}{x^2} + g_{Z_0}$$

Then the signature operator D_g near ∂X looks like

$$D_g = \begin{pmatrix} \frac{1}{x}\eth^Z + D_{Y_0} + xR & -\frac{\partial}{\partial x} - \left(\frac{f-N}{x}\right) \\ \frac{\partial}{\partial x} + \frac{N}{x} & -\frac{1}{x}\eth^Z - D_{Y_0} - xR \end{pmatrix}$$

acting on $\Omega^*(\partial X) \oplus dx \wedge \Omega^*(\partial X)$.

Then $x D_g$ is an <u>elliptic edge operator</u> à la Mazzeo.
The <u>edge calculus</u> is the generalization of the b-calculus to this sort of geometry.
Differential edge operators $\mathrm{Diff}_e^*(X)$ are locally generated by

$$\{x\partial_x, x\partial_{y_1}, \dots, x\partial_{y_b}, \partial_{z_1}, \dots, \partial_{z_f}\}$$

Then from (Hunsicker - Mazzeo), essential self-adjointness & Fredholmness

The problem is that this paper relies heavily on the edge calculus, in particular the composition formula, and there is no analogous calculus for higher rank Witt spaces (ie with more general singularities).
So the goal of project with Albin, Leichtmann, Mazzeo was to give a proof that doesn't use edge calc.

"<u>Thm</u>" (Albin, Leichtmann, Mazzeo, Piazza) Let X be a Witt space endowed with a conic-type metric. Let $r: X \to B\Gamma$ be a classifying map. Then $\exists$ a well-defined index class:

$$\mathrm{Ind}(D_g) \in K_*(C^*_r \Gamma)$$

(item 1 in package)

Progress on other parts so far:

- Need independence of metric (but should not be hard)
- Have it is a bordism invariant for metric bordisms
- Symmetric signatures are defined topologically.
- Need the equivalence of topological & analytic objects (using eg work of Brasselet as in Kasparov)
- Index formula should be complicated - related to work of X. Dai in cone bundle case