MSRI : Topology of stratified spaces 9th Sept. 2008

Sylvain Cappell : Replacement of fixed points of group actions.

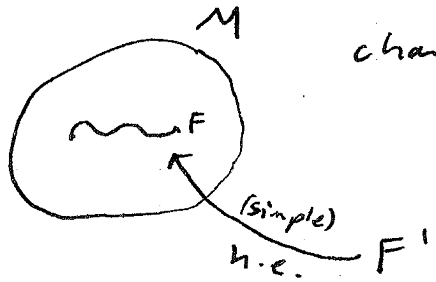
Topological manifold M^n compact Lie group G acting on M
(eg. all finite groups)

action:
locally linear
locally smoothable

Can we "replace" the fixed points of the group action?

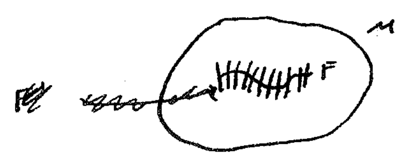
Notation:
F - fixed points
'submanifold

Strong replacement : Can we replace F by F' as the fixed points without changing M, and ~~its isovariant~~ the isovariant homotopy type of the action.



Replacement: can we make F' the fixed point of an action on a manifold isovariantly h.e. to M?
'M'

(Cappell - Min Yan - Weinberger) : Theorem: Assume G-action on nbhd. of 1-skeleton of F ~~has as~~ looks like $2\times$ complex representation of G.



Then one can replace F by any simple h.e. manifold F'.

$$\mathcal{S}^G(M) \xrightarrow{\text{restr.}} \mathcal{S}(F) \quad \text{is a split surj. map.}$$

Isovariant:

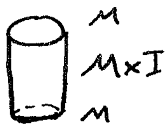
$$X \xrightarrow{f} Y$$

f equivariant : $f(gu) = g\, f(u) \longrightarrow G_u \subset G_{f(u)}$

Isovariant $G_u = G_{f(u)}$ $\to$ strata map to strata

$X = M/G$ $\qquad$ $S(X) \xrightarrow[\text{split surj.}]{} S\left(\begin{smallmatrix}\text{bottom}\\ \text{stratum}\end{smallmatrix}\right)$

This occurs only sometimes, not in general



can't vary one end arbitrarily
(c.f. h-cobordism thm.)

Occurs in supernormal spaces (Cappell-Weinberger)

- simply connected links, even codim

$$S(X) = \prod S^{alg}(\overline{\text{strata}})$$

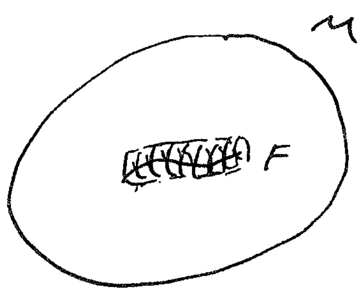
algebraic structure set

- G finite of odd order, small gap hypothesis, smoothable near 1-skeleton of F

  Then we can strongly replace fixed points

- G cyclic, twice complex normal representation, then replacement.

Idea: understand topology near fixed points



E block nbhd. of F

$$\begin{tikzcd}
\dot{E} \arrow[r, phantom, "\subset"] \arrow[d, dashed] \arrow[dr] & E \arrow[d, "\text{block bundle}"] \\
F' \arrow[r, squiggly, "\text{s.h.e.}"'] & F
\end{tikzcd}$$

would like this to be a bundle map.

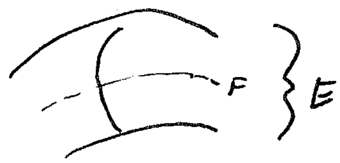
→ Need to replace base of a block ~~bundle~~ fibration.

Then we can form

$$M \setminus \mathrm{int}(E) \cup_{\cancel{\dot{E}}} \text{cylinder}(\dot{E} \to F')$$

For strong replacement we need to keep track of E & the glueing to outside.

→ done by using "bubble quotient" space.



$$\begin{array}{c} E/_{x \sim gx \quad x \in \dot{E}} \\ \big\downarrow \\ F \end{array}$$

Note: for $G = S^1$, these are questions about changing the base of a fibration with fibre Complex Projective Space.

Parity has a qualitative effect. $\times\, \mathbb{C}P^{even} \rightsquigarrow$ iso
$\times\, \mathbb{C}P^{odd} \rightsquigarrow$ zeroes.

leton of F

Browder - Casson - Haefliger(?) - Sullivan - Wall

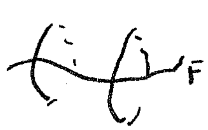
+.

$$Y' \underset{s.h.e.}{\overset{f}{\rightsquigarrow}} Y^n \underset{P.L.}{\overset{i}{\hookrightarrow}} W^{n+k} \qquad k \geq 3$$

$\rightsquigarrow\ i \circ f \sim$ PL embedding.

<u>Example</u>

$G = S^1$, V repr. of S^1 of dim 4



$$F \times D(V)/_{S^1} = F \times D^3$$

~~so can we fiber $F^* \times D^3 \to F'$?~~

$$\begin{array}{ccc} & & F \times D^3 \\ & \swarrow & \downarrow \\ F' & \longrightarrow & F \end{array}$$

obstruction to fibering $F \times D^3$ over F' can be viewed as transfer of obstruction to $F' \to F$ a homeomorphism.

transfer given by mult. by (D^3, S^2) which kills obstructions.

So conclude (1) proof of classical thm.

(2) replace fixed points for this special case.

<u>Lemma:</u> SU_n, ρ the defining rep. of the action.

$$D(2\rho)/U_n = D^3$$

For the argument we view the obstruction to fibration as a transfer, one can generalise this argument to get vanishing.

One can also study question of varying normal rep's of fixed points ~ equiv. to varying the fibres in a block fibration.

$\boxed{\text{Notation: } F\text{–Fibre}}$

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ F' \arrow[u, "s.h.c."] & B \end{tikzcd}$$

One qualitative result: yes often. examples exist for 2^k, p^k cyclic groups