MSRI : Topology of stratified spaces

Clinton McCrory : The weight filtration for real algebraic varieties

(joint work with Parusinsky)

X real algebraic variety

(?)

$H_k(X) = H_k^{BM}(X; \mathbb{Z}/2)$

.)

$0 \subset W_{-k} H_k(X) \subset \cdots W_{-1} H_k(X) \subset W_0 H_k(X) \subset H_k(X)$ filtration by subgroups

the 'weight filtration' (Totaro ICM 2002)

① Filtration is functorial w.r.t algebraic maps

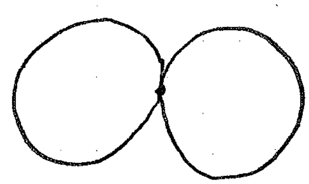
② trivial for compact smooth varieties

$W_{-k} H_k(X) = H_k(X)$

...tt periodicity

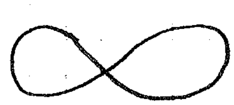
③ Not topological invariant

counterexample

X 

$H_1(X) = W_0 H_1(X) \supset W_{-1} H_1(X) \supset 0$

$H_1(X) = W_{-1} H_1(X)$

X'  $\rightsquigarrow H_1(X') \neq W_{-1} H_1(X')$

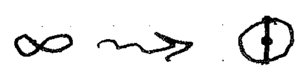
- construction of filtration by resolution of singularities

(Guillén, Navarro Aznar IHES 2002)

weight filtration $\Longleftarrow$ weight spectr. sequence $\Longleftarrow$ weight complex

weight complex: defined up to filtered quasi-iso.

E^r well defined $r \geq 1$

eg  and 

Theorem (McCrory - Parusinski)

To every real algebr. variety X there is assoc. a filtered complex $WC_*(X)$ which is uniquely characterised up to quasi-isomorphism by

① (manifolds)

If X is smooth and projective, $WC_*(X) \underset{q\text{-iso}}{\sim} C_*(X)$

$C_*(X)$ - semi algebraic chain complex with canonical filtration:

$$C_0 \leftarrow C_1 \leftarrow \cdots \quad C_i \leftarrow C_{i+1} \cdots\cdots$$
$$0 \leftarrow 0 \leftarrow \cdots\cdots \quad \ker \partial_i \leftarrow C_{i+1} \leftarrow \cdots$$

② (generalised blow-up)

$$\begin{tikzcd} Y \arrow[r, hook] & X \quad (\text{closed subvariety}) \\ \pi^{-1}(Y) = \tilde{Y} \arrow[u] \arrow[r] & \tilde{X} \arrow[u, "\pi\text{-proper}"'] \end{tikzcd} \qquad \begin{tikzcd} \tilde{X} \setminus \tilde{Y} \arrow[d, "\cong\ \text{iso.}"] \\ X \setminus Y \end{tikzcd}$$

$$\left(\begin{tikzcd} WC_*(\tilde{Y}) \arrow[r] \arrow[d] & WC_*(\tilde{X}) \\ WC_*(Y) & \end{tikzcd} \right) \underset{q.i.}{\cong} WC_*(X)$$

③ (Additivity) $Y \hookrightarrow X$ closed subvariety

$$\big(WC_*(Y) \longrightarrow WC_*(X) \big) \underset{q.i.}{\cong} WC_*(X \setminus Y)$$

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③ gives additivity for the E^1 term of the assoc. spect. seq.

$$\chi\left(E^1_{-i,*}\right) = \beta_i(X) \; - \; i^{\text{th}} \text{ virtual betti number of } X. \qquad (M, P)$$

$$\beta_i(X) = \beta_i(Y) + \beta_i(X \setminus Y)$$

Note: The weight spect. seq. does not collapse.

Real toric varieties - weight complex has purely combinatorial description.

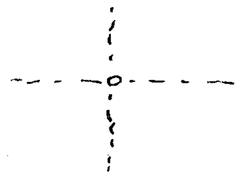
(Bihan, Franz, McCrory, van Hamel)

Local description:

$X = (\mathbb{R}^*)^k$, $C_k(X)$ = "cellular" cell complex | $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$

- cells are the components of X
- 2^k cells defined by signs of coordinates.

nical

eg. $(\mathbb{R}^*)^2$  - 4 components.

think of $\mathbb{R}^*$ as $S^0 \times \mathbb{R}^+$

$$(\mathbb{R}^*)^k = \underbrace{(S^0)^k}_{\text{"discrete torus"}} \times (\mathbb{R}^+)^k$$

$$C_k((\mathbb{R}^*)^k) = C_0((S^0)^k) = \text{group algebra of } (S^0)^k$$

$\mathcal{J} \subset C_0((S^0)^k)$ 'augmentation ideal'

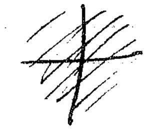
$$\mathcal{J} = \ker\left(C_0((S^0)^k) \longrightarrow \mathbb{Z}/2 \right)$$
$$\Sigma a_i [t^i] = \Sigma a_i$$

$$C_0((S^0)^k) = C_k((S^0)^k) \supset \mathcal{J} \supset \mathcal{J}^2 \supset \mathcal{J}^3 \supset \cdots \mathcal{J}^k \supset 0$$

(filtered) weight complex.

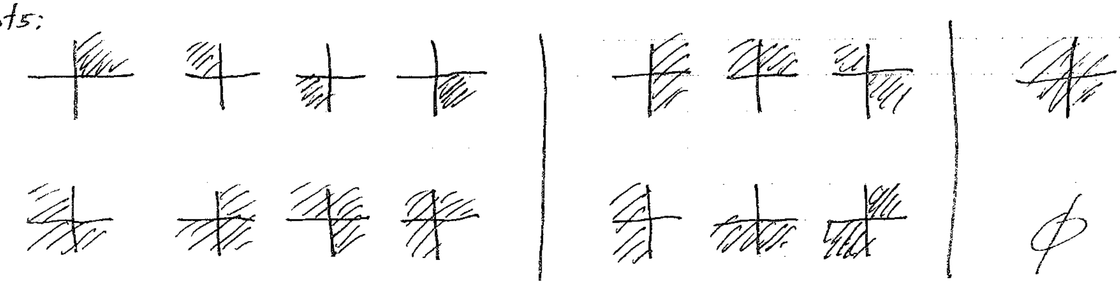
Note $\dim\left(\mathcal{J}^j/\mathcal{J}^{j+1}\right) = \binom{k}{j}$

generators of:

$C_0/\mathcal{J}$  $\mathcal{J}/\mathcal{J}^2$  and  $\mathcal{J}^2$ 

(M, P)

16 elements:



Chain level description of $WC_*(X)$ for an arbitrary real algebraic variety X.

- via Nash constructible functions. $\varphi: X \to \mathbb{Z}$

(conjecture of Bön Pennauch)

$\varphi: X \to \mathbb{Z}$ is <u>Nash constructible</u> $\Leftrightarrow$ $\exists$ maps $f_i : \underset{\substack{| \\ \text{smooth} \\ \text{compact}}}{Z_i} \to X$

$$f_i : Z_i \xrightarrow{\text{proper}} X \qquad \varphi(x) = \sum_i \chi(f_i^{-1}(x))$$

$$\underset{\substack{/ \\ \text{connected} \\ \text{component}}}{Z_i'} \subset Z_i$$

$c \in C_k(X)$ Nash constructible $\Leftrightarrow$ $\exists \varphi: X \to \mathbb{Z}$ Nash constr. $\operatorname{supp}(c) = \{x \mid \varphi(x) \notin 2\mathbb{Z}\}$

- up to a set of dim $< k$.

c is p-Nash constr. $(p \leq 0)$ $\Leftrightarrow$

$$\exists \varphi: X \to 2^{k+p}\mathbb{Z}$$

$$\operatorname{supp}(c) = \{x \mid \varphi(x) \notin 2^{k+p+1}\mathbb{Z}\}$$