MSRI Topology of stratified spaces 9th Sept. '08

Jim Fowler: Lattices with torsion and rational homology manifolds

Question (Wall) $B\Gamma$ P.D $\xrightarrow{?}$ $B\Gamma \simeq$ mfld. "Existence Borel conj."

Conjecture (B,F,M,W) $B\Gamma$ P.D. $\to$ $B\Gamma \simeq \mathbb{Z}$ homology mfld.

Q. (Davis) $B\Gamma$ P.D over Ring R $\xrightarrow{?}$ $B\Gamma \simeq R$-homology mfld.

Γ torsion free.

Related Question: $R = \mathbb{Q}$ Γ-torsion

X space, $\pi_1(X) = \Gamma$ $\tilde{X}$ (univ. cover) — $\mathbb{Q}$ acyclic and $\mathbb{Q}$ homology mfld.

Examples: Γ-uniform lattice.

Γ torsion free $K\backslash G/\Gamma$ $\mathbb{Z}$-PD.

Γ not torsion free $B\Gamma$ $\mathbb{Q}$-P.D.

<k.

$\pi_1(K\backslash G/\Gamma) \neq \Gamma$

$\mathbb{Q}$ homology mfld, singular

Thm: Γ uniform lattice, Γ with torsion (2-torsion?)

$\to$ $\nexists X$ $\pi_1(X) = \Gamma$, $\tilde{X}$ $\mathbb{Q}$-acyclic $\mathbb{Q}$-homology mfld.

Examples exist for Γ (not lattice) with torsion st. $\exists X \ldots$

Finiteness:

Convenience: $\Gamma_0 \lhd \Gamma$ torsion free finite index subgroup with Γ/Γ_0 cycli

$B\Gamma_0 = K\backslash G/\Gamma_0 \circlearrowleft \mathbb{Z}_p$

Q: $\exists X$ $\pi_1(X) = \Gamma$ $\tilde{X}$ $\mathbb{Q}$-acyclic, X finite complex

Lefschetz: $\chi\left(\left(\tilde{X}/\Gamma_0\right)^{\mathbb{Z}_p}\right) = 0$

$\chi\left(B\Gamma_0^{\mathbb{Z}_p}\right) = 0$ -necessary condition.

~~Ne~~ Necessary and sufficient: $\chi\begin{pmatrix}\text{each}\\ \text{connected}\\ \text{comp. of}\\ \text{fixed set}\end{pmatrix} = 0 \rightarrow \chi^{\text{equiv.}} = 0$

This is very doable eg odd dim or cross with S^1.

---

if we had such an $\tilde{X}$:

$$\begin{tikzcd}
 & \sigma_*^{\Gamma}(\underset{\substack{\parallel \\ E\Gamma_0}}{\underline{E}\Gamma}) \\
\sigma_*(\tilde{X}) \arrow[ur, mapsto] & L_n(\mathbb{Q}\Gamma) \arrow[u, Rightarrow] \\
H_n(B\Gamma;\mathbb{L}\mathbb{Q}) \arrow[u, Rightarrow] \arrow[ur, "\text{assembly}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\sigma_*(\tilde{X}) \arrow[r] & \sigma_*^{\Gamma}(\underline{E}\Gamma) \\
H_n(B\Gamma;\mathbb{L}\mathbb{Q}) \arrow[r, "\text{assembly}"] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr] & L_n(\mathbb{Q}\Gamma) \\
H_n(\underline{B}\Gamma; H_n(B\Gamma, \mathbb{L}\mathbb{Q})) \arrow[r] & H_n(\underline{B}\Gamma; \mathbb{L}\Gamma_0) \arrow[u, hook]
\end{tikzcd}$$

so, to prevent existence, make sure $\sigma_*^{\Gamma}(\underline{E}\Gamma)$ not hit by diag. arrow.
