MSRI Topology of Stratified Spaces

Filipp Levikov: The (intersection) homology groups of a fibre bundle over a sphere

$$F \longrightarrow E \longrightarrow B \qquad \text{Fibre bundle.}$$

Question: $H_n(E)$?

Normally Leray Serre Spect. Seq.:

for special B, F $\rightsquigarrow$ Long exact seq.

$F = S^n \rightsquigarrow$ Gysin sequence
$B = S^n \longrightarrow$ Wang sequence

istance,
not
row.

Given $F \to E \to B$ with F PL strat pseudomfld
s.t. E is also PL-strat pseudomfld.

Q: what is $IH_i^{\bar{p}}(E)$?

$$IH_i^{\bar{p}}(F) \xrightarrow{i_*} IH_i^{\bar{p}}(E) \longrightarrow IH_{i-n}^{\bar{p}} \cdot \xrightarrow{\partial}$$

"Intersection homology Wang sequence"

$\bar{p} = 0 \rightsquigarrow \mathbb{I}C^{\bar{p}}_{F/E} \simeq \mathbb{R}_{F/E}$ Hypercohomology yields Wang sequence.

Spect. Seq.? Yes

Given $\mathbb{A}^\cdot$ over X

$$E_2^{p,q} = H^p(X, \mathbb{H}^q(\mathbb{A})) \Longrightarrow \mathcal{H}^{p+q}(X; \mathbb{A}^\cdot)$$

so, $F \to E \longrightarrow B$ regard $R\pi_* \mathbb{I}C^{\bar{p}}_E$ over $X = S^n$

$$E_2^{p,q} = H^p(S^n; \mathbb{H}^q R\pi_* \mathbb{I}C^\cdot_E) \Longrightarrow IH^{\bar{p}}(E)$$

for $n \geq 2$ sequence collapses at second stage.

(Leray sheaf becomes constant, stalk $IH_{-p-n}(F)$)

Induced maps: $f : Y \longrightarrow X$

(i) *Normally nonsingular injection* of codim c

Y sits in c-dim VB nbhd.

$f^* \mathbb{I}C_X \cong \mathbb{I}C_Y[c]$

(ii) *Norm. non sing. projection* of codim c

$f^{-1}(x)$ is submfld of dim $-c$

(iii) <u>Normally non-singular</u> factorises as (i) (ii)

$j : b_0 \times F \hookrightarrow E$ $\qquad$ $i : U \times F \hookrightarrow E$ incl. of complement

dist. triangle $\quad Rj_! j^! \mathbb{I}C^{\bar p}_E \longrightarrow \mathbb{I}C^{\bar p}_E \longrightarrow Ri_* i^* \mathbb{I}C^{\bar p}_E \xrightarrow{[1]}$

apply hypercohomology

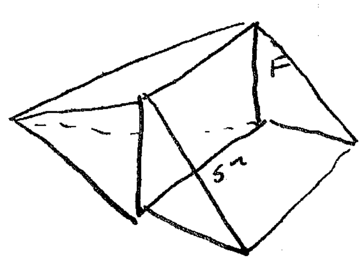
$\mathbb{I}C_F \qquad\qquad \mathbb{I}C^{\bar p}_{U\times F}$

$$IH^{\bar p}_i(F) \xrightarrow{j_*} IH^{\bar p}_i(E) \xrightarrow{i^*} IH^{\bar p}_i(U\times F) \cong IH^{\bar p}_{i-n}(F)$$

– Wang sequence

$$\begin{array}{ccc} b_1 \times F & \overset{\ell}{\hookrightarrow} & U\times F \cong \mathbb{R}^n \times F \\ & \searrow^{k} & \downarrow i \\ & & E \end{array}$$

Application: $\quad M = cyl(\pi) \cup cone(E)$ $\quad$ generalizes Thom space



for $n+k+1$ even and $\quad IH^{\bar m}_{\frac{n+k+1}{2}}(E) \longrightarrow IH^{\bar m}_{\frac{n+k+1}{2}}(F)$ surj.

$\Rightarrow IH^{\bar m}_{n+k+1/2}(M) = 0$

hence, if $n+k+1 \equiv 0\ (4)$ $\rightsquigarrow \underline{\sigma(M) = 0}$

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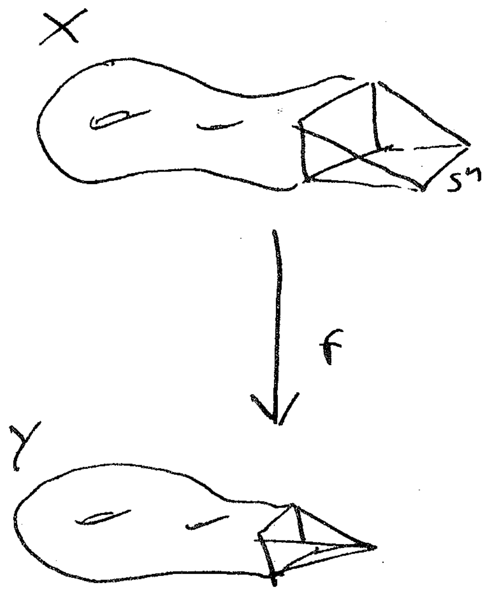
given $f: X^m \to Y^m$ stratified map between (whitney stratified) pseudomflds.

$$\sigma(X) = \sigma(Y) + \sum_{\substack{\text{lower dim}\\ \text{strata}}} \sigma(V)\,\sigma(M_V) \qquad \pi_1(V) = 0$$



$$\sigma(X) = \sigma(Y) + \sigma(M)$$

above assumptions

$\rightsquigarrow \sigma(X) = \sigma(Y)$