MSRI: Topology of singular spaces 10th Sept. '08

Anatoly Libgober: Elliptic genus of singular varieties

(McPherson '83): Which chern numbers of singular algebraic varieties can be defined so that if $X' \to X$ is a small resolution, then chern number of X = chern number of X'?

everything complex $\mathbb{C}$

$\left(\pi : X' \to X \text{ is small} \Leftrightarrow \operatorname{codim}\{x \in X \mid \dim \pi^{-1}(x) = i\} \geq 2i \right)$

$$\begin{tikzcd} X' \arrow[dr] & & X'' \arrow[dl] \\ & X & \end{tikzcd}$$

Ex. Todd genus is comb. of chern monomials

For 5 folds $c_2 c_3$ depends on small resolution

Totaro (2000): Singular spaces for which the singular locus is a union of nonsingular manifolds of codim 3, such that the normal direction of looks like 3-dim node $xy = zu \subset \mathbb{C}^4$

- can resolve this singularity by blowing up: replace singularity by $\mathbb{P}^1 \times \mathbb{P}^1$
- can blow down one of the $\mathbb{P}^1$'s and get two resolutions with except. set $\mathbb{P}^1$.

unitary bordism

$$\Omega^u \otimes \mathbb{Q} \Big/ \begin{matrix}\text{Ideal gen.} \\ \text{by } X_1 - X_2\end{matrix} \text{ — where } X_1, X_2 \text{ differ by flop.}$$

it turns out that this $= \mathbb{Q}[x_1, x_2, x_3, x_4]$

Image of M in this quotient is elliptic genus of M.

Problem: (1) Is the elliptic genus independent of small resolution?

(2) Does the procedure for defining chern numbers work for more complicated ~~isolated~~ singularities.

Answer (Bonsor, Libgober): Yes, elliptic genus is indep of small resolution and can replace small by crepant.

X - singular algebraic variety, $\mathbb{Q}$-Gorenstein

consider resolution $\pi : X' \to X$

$$\rightsquigarrow \pi^* K_X = K_{X'} + \sum \alpha_i E_i$$

pull back of canonical class

Resolution is <u>crepant</u> $\Leftrightarrow \alpha_i = 0$

Singularity called <u>logterminal</u> if there is a resolution s.t. $\alpha_i > -1$

Invariant independents of resolution can be defined for $\mathbb{Q}$-Gorenstein, log-terminal resolutions:

One can associate to the <u>pair</u> (X, D) the inv. $\mathcal{E}ll(X, D)$

$\rightsquigarrow$ leads to invar. indep. of resolution.

<u>Formula for elliptic genus</u>:

E bundle over X $\to \Lambda_t E := \sum t^i \Lambda^i E \in K(X)[t]$

$$S_t E := \sum t^i \, Sym^i E \in K(X)[[t]]$$

eg. for line bundle L:

$$\Lambda_t L = 1 + tL$$

$$S_t L = 1 + tL + t^2 L^2 + \dots$$

$$= \frac{1}{1-t}$$

<u>Def</u>: $\mathcal{E}ll(X) = \chi \left(\bigotimes_{n \geq 1} \left(\Lambda_{-yq^{n-1}} \Omega^1 \right) \otimes \left(\bigotimes_{n \geq 1} \left(\Lambda_{-y^{-1}q^n} T \right) \otimes S_{q^n} \Omega^1 \otimes S_{q^n} T \right) \right)$

- χ of power series in y and q

written in terms of chern roots:

$c(X) = \prod (1 + x_i)$

Riemann-Roch: $\chi(V) = \int_X ch V \, td X$

$$\int_X \prod_{n,i} x_i \frac{(1 - y e^{-x_i})(1 - y q^n e^{-x_i})(1 - y^{-1} q^n e^{x_i})}{(1 - q^n e^{-x_i})(1 - q^n e^{x_i})(1 - e^{-x_i})} = \mathcal{E}ll(X)$$

$$\theta(z,\tau) = q^{1/8}\, 2\sin \pi z \prod (1-q^{e}) \prod (1-q^{e} e^{2\pi i z})(1-q^{e} e^{-2\pi i z})$$

Fin. on $H \times \mathbb{C}$: $q = e^{2\pi i \tau}$

If $y = e^{2\pi i z}$

$$Ell(X) = \int_X \prod_i x_i \frac{\theta(\frac{x_i}{2\pi i} - z_i, \tau)}{\theta(\frac{x_i}{2\pi i}, \tau)}$$

$D = \sum \alpha_i E_i$

$$Ell(X,D) = \text{sing. exp. in } x_i\ E_i \quad \int x_i \frac{\theta(\frac{x_i}{2\pi i} - z, \tau)}{\theta(\frac{x_i}{2\pi i})} \frac{\theta(\dots)(\dots)}{\theta(\dots)(\,)}$$

<u>Corr</u>: If X is Calabi-Yau $\to$

$Ell(X)$ is Jacobi form.

<u>Elliptic genus and loop spaces</u>

Witten's heuristic argument:

Elliptic genus of X is character valued Lefschetz number for S^1 action on $\mathcal{L}X$ for equivariant bundle $\Lambda_y \Omega^1_{\mathcal{L}X}$

Finite dimensional case: M singular manifold

S^1 acts ~~on~~ V equiv. bundle $g \in S^1$

$$L(g, V) = \sum_i (-1)^i \operatorname{tr}(g\ H^i(V))$$

– this is combination of characters

Atiyah-Bott:

$$\sum_{M^g} \int_{M^g} \frac{ch(V)(g)}{1-e^{-x_j; \theta_j}}\, td\, M^g$$

N_θ = S^1 acts as $e^{i\theta}$

$ch(V)(g) = ch\, V_\theta\, e^{i\theta}$

$$X \subset \mathcal{L}X$$
$$\overset{\shortparallel}{Map(S^1, X)}$$

$t\,\xi(x) = \xi(x-t)$ – X is fixed points of this action

Small loops look like $a_0 + \sum a_n \cos nx + b_n \sin nx$, a_n, b_n tangent vector fields

$$T_{\mathcal{L}X}|_X = T_X \oplus \bigoplus_{n\neq 0} \underbrace{(T_X \oplus T_X)}_{T_X \otimes \mathbb{C}}$$

action of S^1 on nth summand is q^n where $q = e^{2\pi i t}$

X complex manifold

contribution of normal bundle into Lefschetz formula

$$\oplus T_X \otimes \mathbb{C} = \oplus T_X \oplus \Omega^1_X$$

$$\int \frac{x_i (1-y e^{-x_i}) \prod (1-y q^n e^{x_i})(1-y^{-1} q^n e^{-x_i})}{(1-q^n e^{x_i})(1-q^n e^{-x_i})(1-e^{-x_i})}$$

$\Lambda_y \Omega^1_{\mathcal{L}X}$

$\Lambda_y E = \Lambda_y(\oplus L_i) = \Lambda(y E_i)$

$-\Omega^1$ comes with factor y
T comes with y^{-1}

Kofranov-Vasserot $\to$ X ..... $\mathcal{L}X$ formal loop space

... There is chiral de Rahm complex, sheaf of $\mathcal{L}X$, action of formal group of S_m...

Thm: Elliptic genus is Lefschetz number of action

Other applications:

(Borisov, L) (1) Orbifold elliptic genus $\to$ Mc Kay corresp. for ell. genus.

(2) Higher elliptic genus gener. Novikov higher signature.
inv. of K-equiv.

(Waelder, ...) (3) Equivariant elliptic genera.

(4) For surface $\exists$ ext. of ell. genus to more complicated than log terminal.

(5) relation to Vafa-Gopakumar conj., Nekrasov conj.