Sandor Kovacs - Singularities in the Minimal Model Program

Work over $\mathbb{C}$ / X projective

classify alg. varieties
- birational(ly equiv.) | def
  X, Y birational if
  $\exists U \subseteq X, V \subseteq Y$ open dense (Zariski top.)
  s.t. $U \simeq V$

Plan
1) Choose a "nice" representative of each birat. equiv. class
2) Algorithm to get the nice repr.
3) Classify the "nice" guys

What is "nice"? projective smooth?

Start with 3

discrete invariants: dim, genus, degree,

after fixing enough discrete invs $\rightsquigarrow$ moduli theory

Ex: $X \subseteq \mathbb{P}^2$ plane curves
$d = \deg X$ these are parametrized by
$\mathbb{P}^{\frac{d(d-3)}{2}}$

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ line, deg = 1
$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ conic, deg = 2

! degree not intrinsic, depends on the embedding

Q: Is there a natural way to embed X?

$$X \underset{\text{emb.}}{\hookrightarrow} \mathbb{P}^N \quad \overset{\text{roughly}}{\longleftrightarrow} \quad \text{an ample line bdle on } X$$

Q: How do we find line bdles?

Take T_X tangent bdle $\rightsquigarrow \det T_X$

Ω_X cotangent bdle $\rightsquigarrow \det \Omega_X = \omega_X$ (canonical line bdle)

Q: Starting with X, can we find a Y, s.t.

$$X \underset{\text{birat.}}{\sim} Y \quad \text{and } \omega_Y \text{ is ample?}$$

Ex $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$

Euler sequence: $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes(n+1)} \to T_{\mathbb{P}^n} \to 0$

$X \subseteq \mathbb{P}^n$ deg d hypersurface

$\omega_X = \mathcal{O}_X(d-n-1) = \omega_{\mathbb{P}^n}(X)|_X$

ω_X^{-1} ample if $d < n+1$ $\qquad \kappa < 0$

$\omega_X \simeq \mathcal{O}_X$ if $d = n+1$ $\qquad \kappa = 0$

restrict to this case | ω_X ample if $d > n+1$ $\qquad \kappa = \dim$

κ Kodaira dimension: $\kappa = -\infty$ or $0 \le \kappa \le \dim X$

X is of general type: Kodaira dim is max.
$\kappa = \dim$

$X \subset \mathbb{P}^3$, $X = (x^5+y^5+z^5+w^5=0)$, ω_X ample

$\mathbb{Z}/2$ action $\quad x \leftrightarrow y,\ z \leftrightarrow w$

$\rightsquigarrow X/\mathbb{Z}_2$ singular

5 singularities of A_1 type

$Y \xrightarrow[\text{resol.}]{\text{bir.}} X/\mathbb{Z}/2$

$\exists$ 5 (-2) curves
$E \subset Y$, $E \cong \mathbb{P}^1$, $E^2 = -2$

$$\left.\begin{array}{l} \underbrace{\omega_E}_{\mathcal{O}_{\mathbb{P}^1}(-2)} = \omega_Y \underbrace{(E)|_E}_{\mathcal{O}_{\mathbb{P}^1}(-2)} \end{array}\right\} \Rightarrow \omega_Y|_E \cong \mathcal{O}_E$$

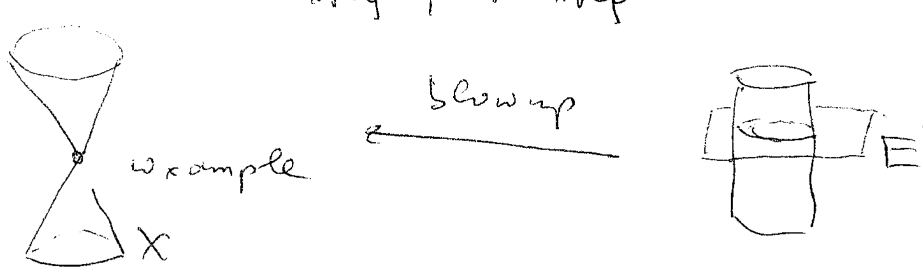
$\omega_{X/\mathbb{Z}/2}$ is ample

(! canonical divisor can be defined for normal varieties !)

X singular, $X_{reg} = X \setminus \mathrm{Sing}\, X \overset{\iota}{\hookrightarrow} X$

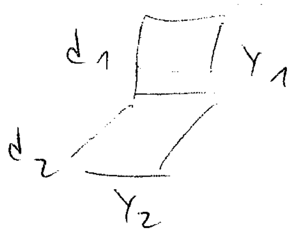
if $\mathrm{codim}(\mathrm{Sing}\, X, X) \geq 2$

$\omega_{X_{reg}}$, $\iota_* \omega_{X_{reg}} =: \omega_X$



Q: Is $\omega_{\hat{X}} + E$ still ample?

$\mathbb{P}^3 \supseteq$



transv. intersection of surfaces (normal crossing)

$Y = Y_1 \cup Y_2$

$\omega_Y|_{Y_1} \simeq \omega_{Y_1}(Y_2|_{Y_1})$

$$\omega_Y|_{Y_2} \simeq \omega_{Y_2}(Y_1|_{Y_2})$$

$$\underset{d_1+d_2-4}{\omega_Y|_{Y_2}} \simeq \underset{d_2-4}{\omega_{Y_2}}(\underset{d_1}{Y_1}|_{Y_2})$$

$$\omega_{\hat{X}} + E|_{\hat{X}} \simeq \omega_{\hat{X}}(E|_{\hat{X}})$$

$$\omega_{\hat{X}} + E|_E \simeq \omega_E(\hat{X}|_E)$$

$$\omega_{\hat{X}}(E|_{\hat{X}})$$

$$\omega_E(\underbrace{\hat{X}|_E}_{\text{! deg d curve}}) \simeq \mathcal{O}_{\mathbb{P}^2}(d-3)$$

ample $\Leftrightarrow$ $d > 3$

$\mathcal{O}_{\mathbb{P}^2}(-3)$ in $\mathbb{P}^2 = E$