Yokura

Motivic characteristic classes of singular varieties

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1 Preliminary ("motif" = motivation)

A "super" general. of counting points of a finite set

X finite

$c(X) = |X|$ the # of elements of X

Satisfies

① $X \cong X' \Rightarrow c(X) = c(X')$

② $c(X) = c(X \setminus Y) + c(Y)$ "Scissor" relation

③ $c(X \times Y) = c(X)\,c(Y)$

④ $c(pt) = 1$

Consider the same thing on TOP

Suppose $\exists\ c_{top}(X) \in \mathbb{C}$ such that X: top space

top "counting"

① $X \underset{\text{top iso}}{\cong} X' \Rightarrow c_{top}(X) = c_{top}(X')$

② $c_{top}(X) = c_{top}(X \setminus Y) + c_{top}(Y)$, $Y \subset X$

③ $c_{top}(X \times Y) = c_{top}(X)\, c_{top}(Y)$

④ $c_{top}(pt) = 1$

<u>Observation:</u> If such a c_{top} exists, then

$$\underline{c_{top}(\mathbb{R}^1) = -1} \quad (\text{use } ③)$$

therefore $c_{top}(\mathbb{R}^n) = (-1)^n$

therefore for a finite CW-complex

$$c_{top}(X) = \chi(X) \quad \text{Euler char. of } X$$

(w. the comp. supp.)

To show the existence of $c_{top} = \chi$

we use <u>ordinary homology theory</u>

Consider the simplicial counting on $\mathcal{V}ar$

complex alg. varieties

(1) $X \underset{\substack{alg\\iso}}{\cong} X' \Rightarrow C_{alg}(X) = C_{alg}(X')$

(2) $C_{alg}(X) = C_{alg}(X \setminus Y) + C_{alg}(Y)$, $Y \subset X$ closed subvariety

(3) $C_{alg}(X \times Y) = C_{alg}(X) \cdot C_{alg}(Y)$

(4) $C_{alg}(pt) = 1$

Suppose $\exists$ such a C_{alg}

Let ~~C_{alg}~~ $y = -C_{alg}(\mathbb{C}^1)$, then $C_{alg}(\mathbb{C}^n) = (-y)^n$

Then we get

$$C_{alg}(\mathbb{P}^n) = 1 + C_{alg}(\mathbb{C}^1) + \cdots + C_{alg}(\mathbb{C}^n)$$

$$= 1 + (-y) + y^2 + \cdots + (-y)^n$$

In fact, using Mixed Hodge Structure,

let $uv = -y$, then $E(X, -u, -v) = \chi_{u,v}(X)$

$$= \sum (-1)^i (-1)^{p+q} \dim_{\mathbb{C}} (G_F^p G^W_{p+q} (H_c^i(X, \mathbb{C}) u^p v$$

"cobordism" counting

$$c_{cob}(X) \in \mathbb{C} \quad (\text{genus})$$

① $X \underset{\text{cobordant}}{\cong} X' \Longrightarrow c_{cob}(X) = c_{cob}(X')$

② $c_{cob}(X \sqcup Y) = c_{cob}(X) + c_{cob}(Y)$

③ $c_{cob}(X \times Y) = c_{cob}(X)\, c_{cob}(Y)$

④ $c_{cob}(pt) = 1$

2. Motivic char. classes

Let $\mathcal{V}$ be the free abel. gr. generated by isom. cl. of varieties

$$\chi_{u,v} : \mathrm{Iso}(\mathcal{V}) \longrightarrow \mathbb{Z}[u,v]$$

$$[X] \longmapsto \chi_{u,v}(X)$$

, extend lin. homom. on the subgroup generated by

$$S: \quad [X] - [Y] - [X \setminus Y]$$

$\chi_{u,v}(S) = 0$ Then we can cons. $\chi_{u,v}$

$$\chi_{u,v} \quad Iso(Var)/_{\cong} \longrightarrow \mathbb{Z}[u,v]$$

$$\| $$

$K_0(Var)$ the Grothendieck ring of varieties

<u>Want:</u> Can we get ? (functor)

$$? : \; ?(X) \longrightarrow H_*(X) \otimes \mathbb{Z}[u,v]$$

nat. transf. covariant functor

s.t. for $X = pt$, $?(pt) = K_0(Var)$

$$\chi_{u,v} = ?_*(pt) \; ?(pt) = K_0(Var) \longrightarrow \mathbb{Z}[u,v]$$

One answer for ? is the relative Grothendieck group Var

$K_0(Var/X)$ is the free abelian group gen. by isom. classes of

$$[Y \longrightarrow X]$$

modulo the subgroup generated by the following form

$$[Y \xrightarrow{h} X] - [Z \xrightarrow{h|Z} X] - [Y \setminus Z \xrightarrow{h|Y-Z} X]$$

$Z \subset Y$ closed

So $K_0(Var/pt) = K_0(Var)$

$K_0(Var/X)$ is covariantly functorial

$f : X \to X'$

$f_* : K_0(Var/X) \to K_0(Var/X')$

$$[Y \xrightarrow{h} X] \longmapsto [Y \xrightarrow{f \circ h} X']$$

<u>Theorem</u>: If $\exists$ natural transformation

$T : K_0(Var/X) \to H_*(X) \oplus \mathbb{Z}[u,v]$

s.t. (1)

$uv = -y$

$$T([\mathbb{P}^n \to pt]) = \chi_{u,v}(\mathbb{P}^n) = 1 + (-uv) + (-uv)^2 + \dots + (-uv)^n$$

$\hat{K}_0(V_{ov})$

$$= 1 - y + (-y)^2 + \dots + (-y)^n$$

(2) for X smooth

$$T([X \xrightarrow{id_X} X]) = cl(TX) \cap [X]$$

for a certain multipl. class of vector bdles

Then (1) in $uv = -y$, we must have $u = y$, $v = -1$

(2) $cl = td_y$, the generalized Hirzebruch Todd class

$$td_y(E) = \prod_{i=1}^{\operatorname{rank} E} \left(\frac{\alpha_i(1+y)}{1-e^{\alpha_i(1+y)}} - \alpha_i y \right)$$

$y = -1$ c

$y = 0$ td

$y = 1$ L

α_i Chern roots

(3) Such a transformation is unique

Namely, there exists a unique natural transformation

Hirzebruch class $\quad td_{y*} : K_0(var/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y]$

s.t. for X nonsingular

$$td_{y*}([X \xrightarrow{id_X} X]) = td_y(TX) \cap [X]$$

(4) $X = pt$

$td_{y*} : K_0(V_{0}) \to \mathbb{Q}[y]$ is

equal to the χ_y-genus homomorphism

$$\chi_y(X) = \sum (-1)^i \dim_{\mathbb{C}} (Gr_F^p H_c^i(X, \mathbb{C}))(-y)^p$$

Hodge filtration

$\exists!$ nat transf.

$$\Lambda_y^{\text{mot}} : K_0(Var(X)) \to K_0(MHM(X)) \to G_0(X) \otimes \mathbb{Q}[y]$$

mixed Hodge module

s.t. X smooth

$$\Lambda_y^{\text{mot}}([X \xrightarrow{id} X]) = \lambda_y(\Omega_X) = \sum [\Omega_X^p] y^p$$

$$\widetilde{d_*^{BFM}} : G_0(X) \otimes \mathbb{Q}[y] \to H_*(X) \otimes \mathbb{Q}[y, (1+y)^{-1}]$$

Main part

$$\widetilde{td_*^{BFM}} := \sum \frac{1}{(1+y)^i} (td_*^{BFM})_i$$

Baum-Fulton-MacPherson Riemann Roch $td_*^{BFM} : G_0(X) \to H_*(X) \otimes \mathbb{Q}$

s.t. for X smooth

$$td_{y}^{BFM}(\mathcal{O}_X) = td(X)\,\eta(X)]$$

$y = -1$

$$\begin{tikzcd}
K_0(Var/X) \arrow[rr, "td_{*y}", "td_{-1*}"'] \arrow[dr, "\varepsilon"'] & & H_*(X) \otimes \mathbb{Q} \\
& \mathcal{F}(X) \arrow[ur, "c_* \text{ MacP}"'] & \\
& G_0(X) & \\
& \Omega^X \arrow[uuur, bend right] &
\end{tikzcd}$$

MacPherson Chern class

Aim of [illegible]

CS Sw.

One conjecture

Any natural transf

$$T : K_0(Var/X) \to H_*(X) \otimes \mathbb{Q}[y]$$

is a lin. comb. of

$$T = \sum_{i \geq 0} r_i \left(td_{*y}\right)_i$$

G. Kennedy, C. McCrory, S.Y.
(CRAS 1984)

$$T : F(X) \longrightarrow H_*(X) \otimes \mathbb{Q}$$

is a lin. comb.

$$T = \sum_{i=0} r_i \left(C_{*i}^{Mac} \right)$$

Id-mod

$$\longrightarrow \quad td_{y*}(X) := td_{y*}$$

$$td_{-1*}(X) = c^{Mac}(X) \otimes \mathbb{Q}$$

Hodge-Todd $\quad td_{0*}(X) \neq td^{BFM}(X)$

Todd class $\quad td_{1*}(X) \neq L^{GM}(X)$