The perverse filtration and the Lefschetz Hyperplane Theorem

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1 Mixed Hodge Theory of algebraic varieties

In this section, following a kind request by the organizers of the workshop, I give a (very rough, brief and highly incomplete) outline of the theory of (mixed) Hodge structures on the intersection cohomology of complex algebraic varieties.

Let X be a complex projective manifold.

The *Hodge Decomposition* is a canonical direct sum decomposition of the de Rham/singular cohomology of X:

$$H^j(X,) = \bigoplus_{p+q=j} H^{p,q}(X), \qquad H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

One can view $H^{p,q}(X)$ as harmonic (p,q)-forms.
The *Hodge filtration* is $F^p H^j(X,) = \oplus_{p' \geq p} H^{p', j-p'}(X)$.
This structure, which is part of the data of a so-called *pure Hodge structure of weight* j, is functorial for maps of projective manifolds.

If X is a a complex algebraic variety, then the singular cohomology *does not* admit such a decomposition.
It is an important result of Deligne's [Deligne, Hodge I-II-III] that every complex algebraic variety admits a canonical *mixed* Hodge structure which is functorial for maps of complex algebraic varieties.
Roughly speaking, the mixed Hodge structure on the cohomology of X is given by two filtrations F^* (Hodge) and W_* (weight) such that F induces a Hodge decomposition on the graded pieces of W; we have

$$0 \subseteq W_0 \subseteq \ldots \subseteq W_{2j} = H^j(X,), \qquad \frac{W_l}{W_{l-1}} \otimes \; = \bigoplus_{p+q=l} H^{p,q}(X), \quad \forall 0 \leq l \leq 2j.$$

The existence of this structure was guessed by analogy with the filtrations on cohomology given by the action of Frobenius on the étale cohomology of algebraic varieties over the algebraic closure of finite fields.

The presence of these structures imposes strong restrictions on the topology of algebraic varieties and the maps between them. Here is a striking example:

ssd **Theorem 1.1 (Deligne)** *Let $f : X \to Y$ be a smooth projective map. The monodromy representation on the cohomology of the fibers*

$$\pi_1(Y, y) \longrightarrow Aut(H^*(X_y))$$

is semisimple (direct sum of simple; simple = no sub-representation).

Question 1.2 *What about (mixed) Hodge structures on intersection cohomology?*

Intersection cohomology is a geometric homology theory which is well-suited for the study of complex algebraic varieties. It has been introduced by Goresky-MacPherson.
It is an invariant which has proved itself to be very useful in geometry, representation theory and combinatorics. See [MacPherson, 1983 ICM talk] and the survey [de Cataldo-Migliorini, survey on the Decomposition Theorem, arxiv and to appear in B.A.M.S.].

Here are two important applications:

- Stanley's proof of certain combinatorial identities on polytopes via the properties of intersection cohomology of the associated singular toric varieties, and
- proof that the Kazhdan-Luszitg polynomials for the Weyl group of a semisimple algebraic group G are computed via the intersection cohomology complex of the associated Schubert varieties on G/B.

The following are two key and deep results concerning intersection cohomology:

- the *Decomposition Theorem* of Beilinson-Bernstein-Deligne-Gabber states that given a proper map of varieties, the intersection cohomology of the domain admits a direct sum decomposition into intersection cohomology groups with twisted coefficients defined on the target;
- *the Relative Hard Lefschetz Theorem* is a generalization of the classical Hard Lefschetz Theorem to singular varieties and maps.

Here is a striking consequence of the Decomposition Theorem.

Theorem 1.3 *The intersection cohomology groups of a variety Y always appear as a direct summand of the ordinary cohomology of the resolution of the singularities of Y.*

At this stage (1982), it was natural to conjecture that the intersection cohomology groups should enjoy many of the properties of the cohomology of smooth varieties: the Hodge-Lefschetz theorems and the existence of natural mixed Hodge structures.

This has been achieved by the work of many authors: Beilinson-Bernstein-Deligne-Gabber for the Hard Lefschetz Theorem (they use algebraic geometry in positive characteristic and the notion of *purity*) M. Saito for the pure and mixed Hodge theory (he uses D-modules and the notion of *mixed Hodge modules*).

Some of these results have been re-proved and amplified to include new properties of the refined intersection forms on the fibers of maps by de Cataldo-Migliorini (this approach uses the classical Hodge Theory of projective manifolds).

This ends my brief discussion of the mixed Hodge theory of algebraic varieties.

2 Filtrations in cohomology

The purpose of this talk is to report on recent joint results with L. Migliorini at Bologna, where we give a geometric description of the perverse filtration on the (hypercohomology) groups $H^*(Y, K)$ of a quasi projective variety Y with coefficients in a bounded constructible complex K of sheaves of abelian groups.

The singular cohomology groups $H^j(X,)$ of a topological space X are defined using the complex of singular cochains:

$$H^j(X,) = H^j(C^*(X,)).$$

In this high degree of generality, cohomology does not seem to carry further structure. This is a bit of an exaggeration, think of the relation to homotopy, the ring structure etc.

If, for example, X is a complex projective manifold, then we have the Hodge filtration

$$F^p H^j(X,) = \bigoplus_{p' \geq p} H^{p', j-p'}(X) \subseteq H^j(X,).$$

Typically, by adding structure to X, cohomology finds itself endowed with additional structure, often in the form of a filtration

$$\ldots \subseteq F^{p+1} H^j(X) \subseteq F^p H^j(X) \subseteq \ldots \subseteq H^j(X), \qquad \forall p \in .$$

For example,

- the datum of an open covering $\mathcal{U}$ of X yields the Čech spectral sequence and filtration, or
- the datum of a continuous map $f : X \to Y$ yields the Leray spectral sequence and filtration.

The Hodge, Čech and Leray filtrations are all produced by the same mechanism (introduced by Grothendieck) that starts with a filtered complex of sheaves K on a space Y and produces a spectral sequence

$$E_1^{p,q} = H^{p+q}(Y, Gr_F^p K) \Longrightarrow H^{p+q}(Y, K)$$

abutting to the filtration

$$F^p H^*(Y, K) := \text{Im}\{ H^*(Y, F^p K) \longrightarrow H^*(Y, K). \}$$

Example 2.1 (Hodge-to-de Rham) Let X be a complex manifold of complex dimension n and

$$\Omega_X^\bullet := \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^2 \longrightarrow \ldots \longrightarrow \Omega_X^n \longrightarrow 0$$

be the holomorphic de Rham complex. There is the decreasing "stupid" filtration by subcomplexes of $\Omega^\bullet_X$

$$\sigma_{\geq l}\Omega^\bullet_X := \quad 0 \longrightarrow \Omega^l_X \longrightarrow \ldots \longrightarrow \Omega^n_X \longrightarrow 0.$$

The associated spectral sequence is the Hodge-de Rham spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X) \Longrightarrow H^{p+q}_{dR}(X,).$$

The associated filtration is the *Hodge* filtration (met earlier)

$$\sigma_{\geq p}H^j_{dR}(X,) = \mathrm{Im}\{\, H^j(X, \sigma_{\geq p}\Omega^\bullet_X) \longrightarrow H^j_{dR}(X,)\,\}.$$

If X is complex projective, then the spectral sequence is E_1-degenerate and, moreover, the Hodge filtration has a canonical splitting

$$\sigma_{\geq p}H^j_{dR}(X,) = \bigoplus_{p' \geq p} H^{j-p'}(X, \Omega^{p'}_X).$$

Remark 2.2 When X varies in a family $f : \mathcal{X} \to Y$ the Hodge filtration also varies and this has lead Griffiths to introduce the notion of *Variation of Hodge Structure.*

Example 2.3 (The Leray spectral sequence) Let $f : X \to Y$ be a continuous map and C be a complex of sheaves on X. There is the direct image complex Rf_*C on Y. The cohomology sheaves are called the direct image sheaves R^jf_*C and they are sheaves on Y. If f is proper, then their stalks $(R^jf_*C)_y = H^j(f^{-1}(y), C_{|f^{-1}(y)})$. In general they are the sheaf associated with the presheaf $U \mapsto H^j(f^{-1}(U), C)$.
The direct image complex on Y is filtered by the truncated subcomplexes

$$\tau^pRf_*C := \tau_{\leq -p}Rf_*C \longrightarrow Rf_*C$$

and the Grothendieck mechanism produces the Leray spectral sequence

$$E_1^{p,q} = H^{2p+q}(Y, R^{-p}f_*C) \Longrightarrow H^{p+q}(X, C)$$

with associated Leray filtration. For example if $f : X \to Y$ is a fiber bundle projection and $C = {}_X$, we have, after re-numbering, the perhaps more familiar looking

$$\mathcal{E}_2^{s,t} = H^s(Y, \underline{H}^t(F)) \Longrightarrow H^{s+t}(X,),$$

where $\underline{H}^t(F)$ is the local system of coefficients on Y associated with the fiber F of the bundle:

$$y \longmapsto H^t(F_y,), \qquad \pi_1(Y, y) \longrightarrow \mathrm{Aut}(H^t(F)).$$

Even, when the map is a bundle, the Leray spectral sequence is seldom E_1-degenerate.

Example 2.4 Let $S^3 \to S^2$ be the Hopf map. The only interesting differential in the Leray Spectral sequence is $H^0(S^2,) \to H^2(S^2,)$. Since $b_1(S^3) = 0$ this differential cannot be zero. (In fact, it is cupping with the Euler class of this oriented S^1-bundle, or equivalently, with the first Chern class of the tautological line bundle of $S^2 = {}^1$ (see [Bott-Tu]). This counterexample to degeneration can be promoted to the realm of complex geometry: one deals then with the (compact complex) Hopf surfaces $X := ({}^2)^*/ \longrightarrow {}^1$ (see [Barth-Peters-Van de Ven]).

There is one important case when the Leray sopeoctral sequence is E_1-degenerate

Theorem 2.5 (E_1-degeneration for smooth projective maps) *(Deligne 1968)* *Let $f : X \to Y$ be a proper holomorphic submersion of quasi projective complex manifolds. The Leray spectral sequence for* <u>*rational*</u> *cohomology is E_1-degenerate.*

This is an important property of families of projective manifolds.

By an old result of Ehresmann, f is a differentiable fiber bundle projection. One cannot expect that this fact alone implies the degeneration (e.g. the Hopf map).
It is the Hard Lefschetz Theorem on the fibers of f that implies it.

The degeneration implies a kind of twisted Künneth formula

$$H^j(X,) \simeq \bigoplus_{p+q=j} H^p(Y, R^q f_{*X}).$$

relating the cohomology of X, Y and the fibers in the easiest possible way.
Unlike Künneth, the splitting of the filtration is *not canonical* (hence $\simeq$ instead of $=$).
By not canonical, we mean that there are many splittings and, if you are interested in some property you want to check (e.g. you want the summands to be Hodge substructures), the splitting you choose may not satisfy the wanted property.

Proper submersions are special (Deligne's Theorem). As soon as a map is no longer proper or smooth (or netiher) the spectral sequence ceases to be E_1-degenerate. Still, the spectral sequence and the filtration are fundamental invariants of the map. In the cases when the map is a resolution of singularities, or it is a degeneration of a manifold in a family, they capture some of the features that distinguish a variety from its resolutions/deformations.

q1 **Question 2.6**

1. *Can we describe the Leray spectral sequence geometrically?*
2. *Are there other interesting spectral sequences and filtrations and can we describe them geometrically?*

By work of Arapura, the answer to the first question is affirmative.
The aim of this talk is to answer the second question in the affirmative.

3 A topological analogue and Arapura's result

Let Y be a finite n-dim'l cell complex with k-scheleta $Y_k \subseteq Y$. Then

$$H^p(Y) = H^p\,[\,H^0(Y_0,\emptyset) \stackrel{\partial}{\to} H^1(Y_1,Y_0) \stackrel{\partial}{\to} \ldots \stackrel{\partial}{\to} H^n(Y_n,Y_{n-1})\,]$$

where ∂ are the coboundaries for the triples (Y_q, Y_{q-1}, Y_{q-2}).

Let $f : X \to Y$ be a fiber bundle. We set $X_p =: f^{-1}Y_p$. There is the classical spectral sequence for the filtered space X_* (see [Spanier])

$$E_1^{p,q}(X_p, X_{p-1}) \Longrightarrow H^{p+q}(X)$$

with geometric (= *given by restriction maps to the pre-image of the scheleta*) filtration

$$F^p_{X_*}H(X) = \mathrm{Ker}\,\{H(X) \longrightarrow H(X_{p-1})\}.$$

It is a fact, that

$$E_2^{pq} = H^p(Y, R^q f_*)$$

i.e. *the spectral sequence for X_* "is" the Leray SS* and (up to renumbering):

on $H(X)$: $\boxed{\text{Leray filtration} = \text{filtration associated with } X_*}$.

The key point is that Y_* must be chosen to satisfy

$$H^{s\neq p}(Y_p, Y_{p-1}, R^q f_*) = 0$$

i.e. that Y_* is *cellular* for the direct image sheaves.

We have just seen that the Leray spectral sequence and filtration of a fiber bundle admit a geometric description via a choice of a suitable filtration of the target space that "trivializes" the cohomology of the direct image sheaves.

In general, algebraic varieties *do not* admit a structure of CW complex with scheleta given by algebraic subvarieties.
These would be desirable to stay in the category of algebraic varieties and deduce, for example, that the restriction maps are maps of mixed Hodge structures. This would imply that the Leray filtration is compatible with mixed Hodge structures (MHS) by invoking the most basic functoriality properties of MHS:
the Leray filtration would be identified with the filtration given by the kernels of restriction maps which are sub-MHS by functoriality.

[Arapura 2005] has shown that, for proper maps of quasi projective varieties $f : X \to Y$, one can achieve a geometric description of the Leray spectral sequence and filtration using a filtration by *closed algebraic subvarieties* (of the domain X, if Y is affine, otherwise of an auxiliary variety). Arapura's result answers the first Question **??**.

Theorem 3.1 (Arapura) *Let $f : X \to Y$ be a proper map with Y affine. Then there is a filtration X_* of X into closed algebraic subvarieties such that*

$$\text{on } H(X): \qquad \boxed{\text{Leray filtration} = \text{filtration associated with } X_*}.$$

4 The perverse spectral sequence and filtration

Inspired by Arapura's nice result, L. Migliorini and I give a geometric description of the perverse spectral sequence and filtration [2008 arxiv].

Let me explain the set-up.

Let Y be a complex quasi projective variety and K be a constructible bounded complex of sheaves of abelian groups on Y.
The complex K admits the *standard filtration* using the truncation functors

$$\tau_{\leq p} K \longrightarrow K \quad \text{(genuine subcomplexes).}$$

This yields the *Grothendieck spectral sequence*

$$E_1^{p,q} = H^{2p+q}(Y, \mathcal{H}^{-p}(K)) \Longrightarrow H^*(Y, K).$$

If $f : X \to Y$ is a map and C is on X, then the Grothendieck spectral sequence for $K := Rf_* C$ is the Leray spectral sequence for $H^*(X, C)$ considered earlier.

The theory or (middle) perverse sheaves (see [Faisceaux Pervers]) yields a systems of "perverse truncation" maps

$$\tau_{\leq p} K \longrightarrow K \quad \text{(not subcomplexes).}$$

I do not attempt to define anything "perverse" here.
Let me just say:

- this system of maps yields a perverse spectral sequence (with its perverse filtration)

  $$E_1^{p,q} = H^{2p+q}(Y, \mathcal{H}^{-p}(K)) \Longrightarrow H^*(Y, K),$$

  as well as, in the special case when $K = Rf_* C$, a perverse Leray spectral sequence (with its perverse Leray filtration) and
- that the following result may convince you that this perverse formalism is interesting:

Theorem 4.1 Beilisnon-Bernstein-Deligne-Gabber's Decomposition Theorem *Let $f : X \to Y$ be a proper map. The perverse Leray spectral sequence for the intersection cohomology groups $IH^*(X,)$ is E_1-degenerate.*

This is an important result, with applications to geometry, combinatorics and representation theory.
Note that while the map must be proper, the singularities of the map are unimportant.
The same result is false for the Leray spectral sequence (unless the map is smooth and proper (Deligne)).

Note however, that if $C \neq IC_X$ (the intersection complex of X the degeneration may fail.

Even in the degenerate case, Question **??** remains: is there a geometric description of the perverse (Leray) spectral sequences and perverse (Leray) filtrations?

In what follows, I make no attempt of keeping track of indices. All the statements I make are true up to re-numbering.

decmig **Theorem 4.2 (de Cataldo-Migliorini 2008)** *Let K be a constructible complex on the affine variety Y. There is a filtration Y_* of Y given by general linear sections such that*

$$\text{on } H(Y,K): \quad \boxed{\text{perverse spectral sequence} = Y_* \text{ spectral sequence}},$$

$$\boxed{\text{perverse filtration} = \text{kernels of restriction maps to } Y_*}.$$

Remark 4.3

1. The variant for cohomology with compact supports holds true.
2. The variant with Y quasi projective hodls true, but one needs to consider an additional filtration Z_* of Y by general linear sections (statement omitted).
3. The perverse Leray variant, i.e. for $f : X \to Y$ and C on X, holds true, and the filtration X_* of X is the pre-image of a Y_* on Y (for Y affine; if Y is quasi projective, then, again, we also need the additional filtration Z_*).
4. The map f needs not to be proper.
5. In Arapura's result, the linear sections must be special (containing the bad strata). Here they must be general (transverse to the bad strata).
6. Everything holds for l-adic cohomology.
7. In 2003, we proved that if $f : X \to Y$ is a map of *projective* varieties, then the perverse filtration on $IH^*(X,)$ coincides with the so called weigth filtration associated with the nilpotent action on $IH^*(X,)$ of an (any!) ample line bundle L on Y. It is amusing to note that this description cannot be correct if Y is not projective, e.g. if it is affine, for a line bundle can be ample and trivial.

The proof of Theorem **??** relies on a systematic use of the Lefschetz Hyperplane Theorem for perverse sheaves as a tool to kill all but one relative cohomology group, a techinque imported from the theory of CW complexes (cf. the "cellularity" mentioned earlier).
In fact, the underlying method of proof works equally well in algebraic topology (instead of algebraic geometry) and yields the classical desciption of the Leray Spectral Sequence of a fiber bundle familiar to topologists.

5 First applications

Theorem 5.1 *(M. Saito; uses mixed Hodge modules)*
The perverse Leray spectral sequence for $H(X,)$ and $H_c(X,)$ are of MHS.

Proof. If Y is affine, they coincide with the spectral sequences on X for a filtration X_* on X (as mentioned earlier, if Y is quasi projective, we need the additional filtration). □

This proof using the filtrations Y_*'s is a starting point [de Cataldo, in progress] for proving that the various summands appearing in the Decompostion Theorem (including $IH^*(X,)$ itself!) can be endowed with MHS arising as subquotients of the MHS of suitable smooth varieties.

M. Saito' s theory of MHM had answered this already in the 80's.

A priori, the two resulting systems of MHS, Saito's via mixed Hodge modules and ours via classical Hodge Theory, could be different.
We show they agree.

6 A conjecture on the cohomology of character varieties

Let me discuss a conjecture Migliorini and I have about the cohomology of character varieties.

We hope that our geometric description of the perverse filtration will be useful in establishing the conjecture.

For simplicity, let me discuss only the case of rank two and the reductive group $SL_2()$.

- Σ compact oriented surface of genus g.
- $M = M_B = M_{B,SL_2}^{2d:=2(3g-3)}$ smooth complex **affine** variety, parameterizing representations
$$\pi_1(\Sigma \setminus p) \longrightarrow SL_2, \qquad (\text{loop around } p) \longmapsto -Id.$$
- The cohomology groups $H^k(M)$ have MHS with weight spaces $W_k \subseteq \ldots \subseteq W_{2k}$.
- $M \times \Sigma$ has a complex rank two $\mathcal{C}^\infty$ vector bundle with $c_2 \in H^4(M \times \Sigma)$.
- By looking at the Künneth components of c_2, we obtain
$$\alpha \in H^2(M), \quad \beta \in H^4(M), \quad \Psi_j, \in H^3(M),\ 1 \leq j \leq 2j.$$

[Hausel-Villegas, 2008 arxiv]:

1. The classes above are multiplicative generators of $H^*(M_B)$ and they all have Hodge-weight 4.
2. **Curious Hard Lefschetz**
$$\boxed{\alpha^l : Gr_W^{2d-2l} H^{i-l}(M_B) \stackrel{\simeq}{\longrightarrow} Gr_W^{2d+2l} H^{i+l}(M_B), \qquad \forall l \geq 0}.$$

Note that M_B is affine and α has weight 4. So everything looks quite weird from the point of view of the classical Hard Lefschetz, where one deals with a projective manifold and a positive line bundle (whose Chern class is of Hodge type $(1,1)$ and weight 2).

Pick a complex structure on Σ and call the resulting projective curve C. The representations parameterized by the affine M_B yield flat connections parameterized by a Stein and Hyper-Kähler manifold M_{DR} and the identification "representation=connection" is biholomorphic. Note that in general, M_{DR} is not complex algebraic!

By rotation of the complex structure, we obtain the algebraic M_{Dol}, which parameterizes stable Higgs bundles, i.e. pairs (E, θ), where E is a rank 2 holomorphic vector bundle on C, $\det E = L$, L fixed of degree 1, $\theta : E \to E \otimes \Omega^1_C$, and stability is defined by the condition that every θ-inv subbundle $V \subseteq E$ has $\deg V < \frac{1}{2} \deg E$.

There is a **projective flat map** ($\mathcal{H}$ is for Hitchin, who gave a beautiful interpretation of this map in terms of spectral covers and their Prym's)

$$\mathcal{H} : M_{Dol}^{2(3g-3)} \longrightarrow \Gamma(C, 2K_C) \simeq {}^{3g-3}$$

and α is now $(1,1)$ and in fact $\mathcal{H}$-ample.

In particular, the Relative Hard Lefschetz Theorem (see [Faisceaux Pervers]) holds for the cup-prouct action of α on $H(M_{Dol})$ (statement omitted).

Remark 6.1 $\mathcal{H}$ has been intensively studied by many authors: Laumon, Beilinson-Drinfeld, Kapustin-Witten, Ngo, ...

Though M_{Dol} and M_B are diffeomorphic, and, as we have seen above, the cohomology of $H(M_B)$ is "very" mixed, the a priori MHS $H(M_{Dol})$ is in fact pure!

The diffeomorphism $M_B \simeq M_{Dol}$ yields $H(M_B) = H(M_{Dol})$.

Conjecture 6.2 Under this identification, we should have

$$\boxed{\text{Curious HL} \Longrightarrow \text{Relative HL for } \mathcal{H},}$$

i.e.,

$$\boxed{\text{weight filtration on } H(M_B) = \text{perverse Leray filtration wrt } \mathcal{H} \text{ on } H(M_{Dol})}.$$

We have verified the conjecture for $g = 2$ by using our Theorem **??**.

Mark de Cataldo - The perverse filtration and the Lefschetz Hyperplane Theorem

X projective manifold

$H^j(X,\mathbb{C}) = \bigoplus_{p+q=j} H^{p,q}(X)$ Hodge decomp

Hodge symmetry : $\overline{H^{p,q}} = H^{q,p}$

example of pure Hodge structure weight j

If X is singular, no such thing exists
Still, Deligne discovered that $H(X)$ becomes a Mixed Hodge structure

$F^p H^j(X,\mathbb{C}) = \bigoplus_{p' \geq p} H^{p',q}$ (Hodge filtration)
$\geq p$ dz's

Mixed $H^j(X,\mathbb{C})$ W_* $\mathbb{Q}$-Coeff.
F^* $\mathbb{C}$-Coeff.

$\frac{W_l}{W_{l-1}} \otimes \mathbb{C} = \bigoplus_{p+q=l} H^{p,q}$, $0 \leq l \leq 2j$

Let $f: X \to Y$ proper smooth of quasi-proj. varieties

$\pi_1(Y,y) \to Aut(H^j(F_y))$ (Monodromy reps.)
Semisimple (Deligne '72) (even for singular Y, but smooth fibres)

Mixed Hodge Theory for $IH(Y)$
M. Saito
Homology theory well-suited for $\mathbb{C}$-varieties

Decomposition Thm (BBD)

$f : X \to Y$ proper map
Leray-Spectr. Seq. $\left(\begin{smallmatrix} HX & HF \\ HY & \end{smallmatrix}\right)$
is seldom degenerate

$IH(X) \simeq$ directsum of intersection homology on Y with twisted coeff. (of Y and of subvar.)
↑ middle perv.

all hold for IH — GM, BBDG, M. Saito:
- Hard Lefschetz
- Soft Lefschetz (Hyperplane Section Thm)
- Poincaré Duality
- Hodge Decomp.
- HRBR

MHS (Pure X proj.)

Filtrations in cohomology

$X \qquad 0 \to \mathcal{O}_X \to \cdots \to \Omega_X^n \to$

Grothendieck mechanism $\rightsquigarrow E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C})$

$X \xrightarrow{f} Y$, C complex of sheaves on X

$Rf_* C = K$ complex on Y

$\tau_{\leq i} K \subseteq K$ (standard filtration)

$\rightsquigarrow$ Spectr. Seq. $E_1^{p,q} = H^{2p+q}(Y, R^{-p} f C)$

$\Rightarrow H^{p+q}(X, C)$

Leray Sp. Seq. $\rightsquigarrow$ Leray filtration
(by images of homology of subcomplexes)

Deligne

$f: X \to Y$ proper smooth
then Leray Sp. Seq. is degenerate (twisted Künneth)

Question

1 Can we describe the Leray Sp. Seq. geometrically?

2 Are there other interesting Sp. Seq. and can we describe them geometrically?

1 Yes Arapura

2 Yes joint w/ L. Migliorini

Recall Alg. Top.

Y cell complex, Y_k k-th skeleton

$\rightsquigarrow E_1^{p,q} = H^{p+q}(Y_p, Y_{p-1}) \Rightarrow H^{p+q}(Y)$

$F^p H(Y) = \operatorname{Ker} H(Y) \to H(Y_*)$

$H^0(Y_0, \emptyset) \to H^1(Y_1, Y_0) \to H^2(Y_2, Y_1) \to \ldots$

$X \xrightarrow{f} Y$ fibre bdle, $X_* = f^{-1} Y_*$ $\rightsquigarrow E_1^{p,q} = H^{p+q}(X_p, X_{p-1})$

$\Rightarrow H^{p+q}(X)$

is the Leray Sp Seq.

$L f: X \to Y = \operatorname{Ker} H X \to H Y_*$

Arapura (2005)

$f: X \to Y$ proper (Y affine)

Then $\exists\, Y_* \subseteq Y$ alg. subvariety, $X_* = f^{-1}(Y_*) \subseteq X$

Leray Sp. Seq. = sp. S. f on X_*

$L^* H(X, \mathbb{Z}) = \ker H(X, \mathbb{Z}) \to H(X_*, \mathbb{Z})$

Thm (M. Saito, Arapura) [gap 50 years] The Leray Sp. Seq. is compatible with MHS.

Sheaf	perverse sheaf
$\tau_{\leq i} K \subseteq K$	${}^p\tau_{\leq i} \to K$ (in the derived cat.)
Groth., standard Sp. S., filtration	perverse Sp. S., filtration
constant sheaf $\mathbb{Q}_Y$	IC_Y

$f: X \to Y$ proper. The Leray Sp. S. for $Rf_* IC_X$ is degenerate.

Thm (d-M) Y projective, K complex constructible, bounded

There is $Y_* \subseteq Y$ s.t.

$E_1^{p,q} = H^{p+q}(Y_p, Y_{p-1}) \Rightarrow H^{p+q}(Y, K)$ is perverse Sp. S.