MSRI Topology of stratified spaces 12th Sept. 08

Ludmil Katzarkov: Hodge structures and Homological Mirror symmetry

1. Motivation
2. Homological Mirror symmetry (HMS) definition & examples
3. Applications
4. Some Hodge structures

A-side (symplectic)		B-side (algebraic)
Singularities	$\longleftrightarrow$	classes
Monodromy	$\longleftrightarrow$	cycles
classes	$\longleftrightarrow$	monodromy

1. Motivation

X smooth proj. variety $\dim_{\mathbb{C}} X = n$

$$\mathbb{C}(X) \overset{?}{\underset{?}{\cong}} \mathbb{C}(x_1, \dots, x_n)$$

eg. $X = \mathbb{P}^3$ - rational

$X \subset \mathbb{P}^4$, $\dim_{\mathbb{C}} X = 3$, $\deg X = 4$ - not rational

$X \in \mathbb{P}^4$, $\dim X = 3$, $\deg X = 4$ - not rational

$X = Q_1 \cap Q_2 \subset \mathbb{P}^5$, $\deg Q_i = 2$, $\dim Q_i = 4$ $\to$ rational

$X = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^6$ $\deg Q_i = 2$ $\dim Q_i = 5$ $\to$ not rational

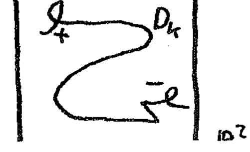
$J(X) \cong H^{2,1}(X)/H^3(X;\mathbb{Z})$ $\quad \overset{\neq}{\text{compare to}}$ $\quad J(C)$ $\quad$ C - Riemann Surface ~~Surface~~

are there examples of 3-dim complex smooth proj. Var. X s.t. $J(X) \cong J(C)$ but X not rational? (Answer unknown.)

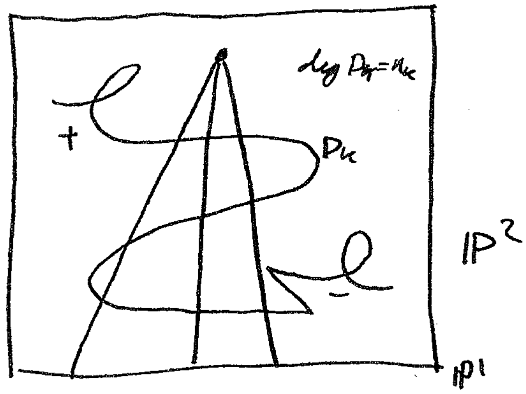
Let $X_{\mathbb{R}}$ be 4-dim symplectic mfld.

Are there examples of 4-dim simply connected symplectic mflds $X_1 \underset{\text{homeo.}}{\cong} X_2$, $GW(X_1) = GW(X_2)$ but not $X_1 \underset{\text{sympl.}}{\cong} X_2$?

X 4 dim sympl. $\underset{KW}{\to}$ $f_k : X \to \mathbb{CP}^2$



p.t.o.



$$f_k : \pi_1(\mathbb{P}^1 \setminus p_1, \dots, p_{e_k}) \to B_r(n_k)$$

<u>B</u>	<u>A</u>
X smooth proj. var. $\mathbb{C}$ $D^b(X) = \begin{cases} \text{Complex } C_i \\ \mathrm{Ext}^j(C_i, C_j) \end{cases}$	X symplectic, ω, $F_{uk} = \begin{cases} L_i, L_j & \text{objects} \\ HF(L_i, L_j) & \text{morphisms} \end{cases}$
$f : X \to \mathbb{C}$ proper $D^b_{sing}(Y, f) = \bigoplus_t D^b(Y_t) / \text{Perfect complexes } \mathrm{Perf}(Y_t)$	$f : Y \to \mathbb{C}$ $\left.\begin{matrix} L_i, L_j \\ HF((L_i, L_j)) \end{matrix}\right\} FS(Y, f)$ ← objects, ← morphisms "∞ category"

<u>HMS conjecture</u> (Kontsevich)

(1) $\forall X$ sm. proj.

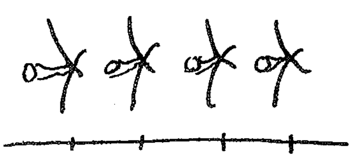
$$\exists f : Y \to \mathbb{C} \quad D^b(X) \cong FS(Y, f)$$

"Landau Ginzburg Model"

(2) $\forall X$ sympl. $\exists f : Y \to \mathbb{C}$ $\quad F_{uk}(X) \cong D^b_{sing}(Y, f)$

<u>Examples</u> $X = \mathbb{P}^3$, $Y' = \mathbb{C}^{*3}$ $\quad f = x + y + z + \frac{1}{xyz}$

$D^b(\mathbb{P}^3) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3) \rangle$



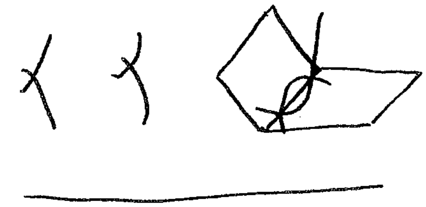
$X = Q_1 \cap Q_2 \subset \mathbb{P}^5$ $\quad \mathbb{C}^{*6}$

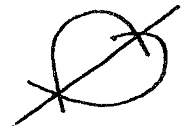
$x_1 \cdot \ldots \cdot x_6 = t$

~~$x_1 t \ldots x_6 = t$~~

$x_1 + x_2 = 0$

$x_3 + x_4 = 0$

$f: Y \to \mathbb{C}$  $Y_1, Y_2, Y_3 \cong \mathbb{F}^2_{p_1,\dots,p_6}$



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<u>Theorem</u> (Orlov)

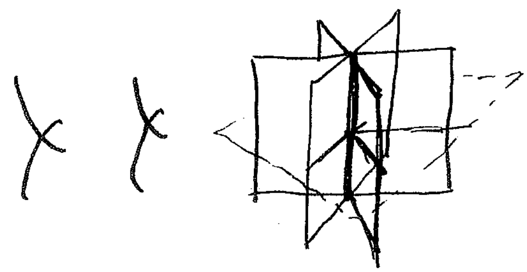
$X \supset C$ $\quad \dim_{\mathbb{C}} X = 3$ $\quad \dim_{\mathbb{C}} C = 1$ $\quad X, C$ smooth

$$D^b(\hat{X}_C) = \langle D^b(X), D^b(C) \rangle$$

<u>Theorem</u> (Abouzaid, Auroux, –)

$$FS(LG(\hat{X}_C)) = \langle FS(LG(X)), FS'(LG(C)) \rangle$$

X 3 dim cubic



"family of K3 ~~surf~~ surfaces"

<u>Theorem</u> Let X be a 3 dim Fano s.t. $\mathrm{rk}\,\mathrm{Pic}(X) = 1$ $\quad X$ - non semi simple

$$X \text{ rational} \longleftrightarrow \begin{array}{c} H^2(Y_t)^{tr} \\ \downarrow \\ 0 \end{array} \quad \text{unipotent}$$

X non rational $\longleftrightarrow$ quasi unipotent

$r: Y_t \longrightarrow Y_0$ $\quad$ retract

$F \in D^b_c(Y_0)$

$$\begin{tikzcd} r^* \mathbb{Z}_{Y_0} & \mathbb{Z}_{Y_t} \arrow[l] \arrow[dl] \\ F \arrow[u] & \end{tikzcd}$$

dist. triangle

<u>Theorem</u> (Cheeses,–) Let X be Fano complete intersection in toric variety

$\dim_{\mathbb{C}} X \leq 4$

$$\mathbb{H}^i(\mathcal{F}) \cong H^i(X)[-1] \circlearrowleft$$

- Similar statement for general types (with shift [1]).

<u>Conjecture</u>

Let X be Fano, $\dim_{\mathbb{C}} X \leq 4$

X complete intersection in toric variety

let $f: Y \longrightarrow \mathbb{C}$ be LG model for X

assume $\exists$ component Z of singular set of $f: Y \longrightarrow \mathbb{C}$ for which the monodromy of $\mathcal{F}|_Z$ is not trivial.

then X is not rational.