MSRI: Topology of stratified spaces 12th September 2008

Gregory Pearlstein: Singularities of Normal Functions

Y = smooth complex projective variety

$$\mu : CH^p_{hom}(Y) \longrightarrow J\left(H^{2p-1}(Y, \mathbb{Z}(p))\right) \quad = Ext^1_{MHS}\left(\mathbb{Z}(0), H^{2p-1}(Y, \mathbb{Z}(p))\right)$$

X smooth complex proj. var. of dimension $2N$.

$L \longrightarrow X$ very ample line bundle

$B = \mathbb{P}H^0(X, L) = |L|$

$$\rightsquigarrow \quad \mathcal{X} = \{ (x, b) \in X \times B \mid b(x) = 0 \}$$

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_B"] & \\ X & & B \end{tikzcd}$$

$\check{X}$ = Dual Variety

$$\mathcal{H}_{\mathbb{Z}} = R^{2N-1}_{\pi_X} \mathbb{Z}(n) \quad \text{over } B' = B - \check{X}$$

$$\mathcal{H} = (\mathcal{H}_{\mathcal{O}}, F^0\mathcal{H}_{\mathcal{O}}, \mathcal{H}_{\mathbb{Z}}) \quad \text{VHS - variation of hodge structures}$$

$$J(\mathcal{H}) \longrightarrow B'$$

Hol. sections: $$J(\mathcal{H})(\mathcal{U}) = \frac{\mathcal{H}_{\mathcal{O}}(\mathcal{U})}{F^0(\mathcal{U}) + \mathcal{H}_{\mathbb{Z}}(\mathcal{U})}$$

A section v of $J(\ )(\mathcal{U})$ is horizontal $\Leftrightarrow \nabla\tilde{v} \subseteq F^{-1} \otimes \Omega^1$

$\zeta \subset H^{N,N}(X, \mathbb{Z})_{prim.}$

$$\longrightarrow v_{\zeta} : B' \longrightarrow J(\mathcal{H})$$

Normal function: $\mathrm{Ext}^1_{VMHS}(\mathbb{Z}(0), \mathcal{H})$

$$0 \to \mathcal{H} \to \mathcal{N} \longrightarrow \mathbb{Z}(0) \longrightarrow 0$$

$$\mathcal{H} \to S \subset \bar{S}$$

$$\begin{array}{c} H^0(S, \mathbb{Z}(0)) \xrightarrow{\partial} H^1(S, \mathcal{H}_{\mathbb{Z}}) \\ \| \\ \mathbb{Z}(0) \end{array}$$

$$d(\nu) = \partial 1 \in H^1(S, \mathcal{H}_{\mathbb{Z}})$$

<u>Theorem</u> (Griffiths/Green)

H. C. $\Leftrightarrow$ $cl_s(\nu)$ not torsion for some ~~[illegible]~~ $|L^k|$ $k \gg 0$

$Z = \{ s \in B' \mid \nu_3(s) = 0 \}$ — prove algebraic, dim > 0

<u>Ex</u>: $C - C^-$ has interesting Z

VMHS from geometry are <u>admissable</u>:

(1) F^p Hodge bundles extend

(2) Existence of the relative weight filtration

<u>Ex</u>.

$$\left.\begin{array}{l} V_{\mathbb{Z}} = \mathbb{Z} e_0 \oplus \mathbb{Z} e \oplus \mathbb{Z} f \\ W_0 = V_{\mathbb{Z}} \quad W_{-1} = \mathbb{Z} e \oplus \mathbb{Z} f \quad W_{-2} = 0 \\ F^1(s) = 0 \quad F^0(s) = \mathbb{C}(e_0 + e^{\frac{1}{s}} f) + \mathbb{C}(e + \frac{1}{2\pi i} \log s f) \end{array}\right| \; VMHS/\Delta^*$$

$$F^{-1}(s) = V_{\mathbb{C}}$$

Monodromy:

$Z = \{ s \in \Delta^* \mid e^{\frac{1}{s}} \in \mathbb{Z} \}$, $N(e_0) = 0$, $N(e) = f$, $N(f) = 0$

<u>Ex</u>. $V_{\mathbb{Z}}$, $W_\cdot$ as above VMHS $\Delta^* \times \Delta$

$N(e_0) = e$, $N(e) = f$, $N(f) = 0$

$$F^1(s) = 0 \qquad F^0(s) = \mathbb{C}\left(e_0 + \left(\tfrac{1}{2\pi i}\log s + \alpha\right) f\right)$$
$$\oplus\ \mathbb{C}\left(e + \left(\tfrac{1}{2\pi i}\log s + \beta\right) f\right)$$

$$F^{-1}(s) = V_{\mathbb{C}} \quad , \alpha, \beta \in \mathcal{O}(\Delta^2) \quad \alpha(0) = \beta(0) = 0$$

$$\rightarrow \quad Z = \{ s \mid \alpha(s) - \beta(s) = 0 \}$$

<u>Conjecture:</u> Let ν be an admissible normal function on a smooth complex algebraic variety S.
Then Z = zero locus of ν, is a complex algebraic subvariety of S.

$\varepsilon \mid k \gg 0$

<u>Theorem:</u> (Brosnan, Pearlstein)
- Conj. true if S is a curve.
- Conj. true if S is a surface.

uses (1-variable) SL_2 orbit theorem

Local Normal Form period map

$$\begin{tikzcd} U^r \arrow[r] \arrow[d] & M \arrow[d] \\ \Delta^{*r} \arrow[r] & \Gamma \backslash M \end{tikzcd}$$

$s_j = e^{2\pi i z_j}$

$$F(z_1, \ldots, z_r) = e^{\sum z_j N_j} e^{\Gamma(s)} . F_\infty$$

$$SL_2 \text{ orbit theorems} \Rightarrow Y_z = e^{\sum z_j N_j} e^{\Gamma(s)} Y_\infty$$

15/D*

f)=0