

Gregory Pearlstein: Singularities of Normal Functions

$Y =$  smooth complex projective variety

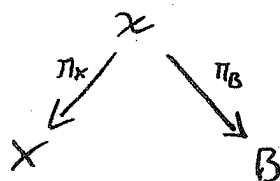
$$M: CH_{\text{hom}}^p(Y) \longrightarrow J(H^{2p-1}(Y, \mathbb{Z}(p))) = \text{Ext}_{\text{MHS}}^1(\mathbb{Z}(p), H^{2p-1}(Y, \mathbb{Z}(p)))$$

$X$  smooth complex proj. var. of dimension  $2N$ .

$L \rightarrow X$  very ample line bundle

$$B = \mathbb{P}H^0(X, L) = |L|$$

$$\rightarrow \mathcal{X} = \{ (x, b) \in X \times B \mid b(x) = 0 \}$$



$\check{X} =$  Dual Variety

$$H_{\mathbb{Z}} = R_{\mathbb{Z}}^{2N-1} \mathbb{Z}(n) \quad \text{over } B' = B - \check{X}$$

$$\mathcal{H} = (\mathcal{H}_0, F^0 \mathcal{H}_0, \mathcal{H}_{\mathbb{Z}}) \quad \text{VHS - variation of hodge structures}$$

$$J(\mathcal{H}) \longrightarrow B'$$

$$\text{Hdg. sections: } J(\mathcal{H})(\mathcal{U}) = \frac{\mathcal{H}_0(\mathcal{U})}{F^0(\mathcal{U}) + \mathcal{H}_{\mathbb{Z}}(\mathcal{U})}$$

A section  $v$  of  $J(\mathcal{H})(\mathcal{U})$  is horizontal  $\Leftrightarrow \nabla v \in F^1 \otimes \mathcal{U}^1$

$$\mathcal{S} \subset H^{M, N}(X, \mathbb{Z}) \text{ prim.}$$

$$\rightarrow v_{\mathcal{S}}: B' \rightarrow J(\mathcal{H})$$

Normal function:  $\text{Ext}_{\text{VMHS}}^1(\mathbb{Z}(0), \mathcal{H})$

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{N} \rightarrow \mathbb{Z}(0) \rightarrow 0$$

$$\mathcal{H} \rightarrow S \subset \bar{S}$$

$$H^0(S, \mathbb{Z}(0)) \xrightarrow{\cong} H^1(S, \mathcal{H}_{\mathbb{Z}})$$

$$d(V) = 21 \in H^1(S, \mathcal{H}_{\mathbb{Z}})$$

Theorem (Griffiths/Green)

H.C.  $\Leftrightarrow d_5(V)$  not torsion for some  $k \gg 0$

$$Z = \{s \in B^1 \mid v_3(s) = 0\}$$

- prove algebraic + dim  $> 0$

Ex:  $C-C^1$  has interesting  $Z$

VMHS from geometry are admissible:

(1)  $\mathbb{F}^p$  Hodge bundles extend

(2) Existence of the relative weight filtration

Ex.  $V_{\mathbb{Z}} = \mathbb{Z}e_0 \oplus \mathbb{Z}e \oplus \mathbb{Z}f$

$$W_0 = V_{\mathbb{Z}} \quad W_{-1} = \mathbb{Z}e \oplus \mathbb{Z}f \quad W_{-2} = 0$$

$$F^1(s) = 0 \quad F^0(s) = \mathbb{C}(e + e^{\frac{1}{2}}f) + \mathbb{C}(e + \frac{1}{2\pi i} \log s f)$$

$$F^{-1}(s) = V_{\mathbb{C}}$$

VMHS / DS

$$Z = \{s \in \Delta^* \mid e^{\frac{1}{2}} \in \mathbb{Z}\}$$

Monodromy:

$$N(e_0) = 0, N(e) = f, N(f) = 0$$

Ex.  $V_{\mathbb{Z}}, W$  as above VMHS  $\Delta^* \times \Delta$

$$N(e_0) = e, N(e) = f, N(f) = 0$$

$$F'(s) = 0 \quad F^0(s) = \phi \left( c_0 + \left( \frac{1}{2\pi i} \log s + \alpha \right) f \right)$$

$$\oplus \phi \left( c + \left( \frac{1}{2\pi i} \log s + \beta \right) f \right)$$

$$F^{-1}(s) = \bigvee_c, \quad \alpha, \beta \in \mathcal{O}(\Delta^2) \quad \alpha(0) = \beta(0) = 0$$

$$\rightarrow Z = \{s \mid \alpha(s) - \beta(s) = 0\}$$

Conjecture: Let  $v$  be an admissible normal function on a smooth complex algebraic variety  $S$ .

Then  $Z =$  zero locus of  $v$ , is a complex algebraic subvariety of  $S$ .

Theorem: (Brosnan, Pearlstein)

- Conj. true if  $S$  is a curve.
- Conj. true if  $S$  is a surface.

← Uses (1-variable)  $SL_2$  orbit theorem

Local Normal Form period map

$$\begin{array}{ccc} U^r \setminus M & \longrightarrow & \mathbb{R}^r \setminus M \\ \downarrow & & \downarrow \\ \Delta^{\text{or}} & \longrightarrow & \mathbb{P}^1 \setminus M \end{array}$$

$$s_j = e^{2\pi i \epsilon_j}$$

$$F(z_1, \dots, z_r) = e^{\sum \epsilon_j M_j} e^{P(s)}, F_{\infty}$$

$$SL_2 \text{ orbit theorem} \rightarrow Y_{\mathbb{Z}} = e^{\sum \epsilon_j M_j} e^{P(s)} Y_{\infty}$$

(5)  $\mathbb{D}^*$

$f=0$