

Jörg Schürmann: Characteristic classes of mixed Hodge modules

related work: Brasselet - Schürmann - Yokawa  
Cappell - Libgober - Maxim - Shameson

- (1) Hodge genera
- (2) Char. classes of VMHS :  $MHS^*$
- (3) Calculus of MHS
- (4) Char classes of MHS :  $MHC_*$

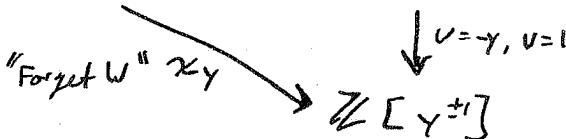
(1)  $MHS^p =$  abelian category of (graded pol.)  $\mathbb{Q}$ -mixed H.S.

- tensor product  $\otimes$ , duality  $( )^\vee$

$(V, F^\bullet, W_\bullet)$  pure of weight  $n \Leftrightarrow g_k^W(V) = 0 \quad k \neq n$

pure weight  $n \rightarrow$  polarisation  $\rightarrow V \xrightarrow{\sim} V^\vee(-n)$

Ring homom.  $K_0(MHS^p) \xrightarrow{E} \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$



Ex. (Deligne)

$X$  alg. /  $\mathbb{C} \quad H_{c,0}^1(X; \mathbb{Q}) \notin MHS^p$

$Y \xrightarrow{d} X \xleftarrow{\text{open}} X \setminus Y$

$\dots \rightarrow H_c^i(Y) \rightarrow H_c^i(X) \rightarrow H_c^i(X \setminus Y) \rightarrow \dots$

$\rightarrow$  ring hom.  $K_0(\text{Var}/\mathbb{C}) \rightarrow K_0(MHS^p)$

$[X] \mapsto [H_c^*(X)]$

Question:  $X$  alg, compact, smooth.

Is there a char. class of vector bundles

$$E(X) = E([H_c^*(X)]) \stackrel{?}{=} \deg(c_1^*(TM) \cap [M])$$

top ch #

Answer: No!

$X, X'$  elliptic curves  $X' \xrightarrow{d:1} X$  cover

top #  $\rightarrow E(X') = d E(X)$  but  $E(K) = (1-u)(1-v) = E(K') \neq 0$

Yes for  $X_Y$  - Hirzebruch '59

(2)  $X$  smooth alg. lc

VMHS = abelian cat. of wt of  $\mathbb{Q}$ -MHS

-  $\otimes$  prod, duality  $( )^\vee$ .

$(L, F^\bullet, W_\bullet)$   $L$  loc. syst. on  $X$ ,  $W_\bullet$  Filtr. on  $L$

$V = L \otimes_{\mathbb{C}} \mathbb{C}_X$  holomorphic VB with  $\nabla = \text{Id}$

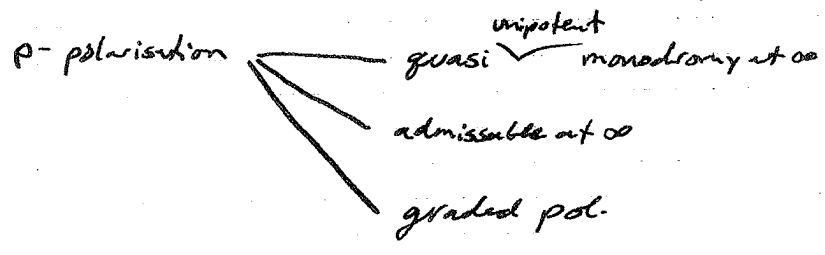
$\nabla \circ \nabla = 0, L = \ker(\nabla)$

$\nabla: V \rightarrow V \otimes \Omega^1_X$

$\begin{matrix} U \\ \mathbb{F}^p \end{matrix} \nabla \rightarrow \mathbb{F}^{p-1} \otimes \Omega^1_X$

Rem:  $V$  pure choice of polarisation  $\rightarrow L \rightarrow L^\vee$   
Poincaré local system.

VMHS<sup>p</sup>



$X \subset \bar{X} \supset D = \bar{X} \setminus X$   
open

DNC

Deligne Ext:  $(\bar{V}, \bar{F}^\bullet, \bar{\nabla}) \xrightarrow{\text{Ext}} \bar{\nabla}: \bar{V} \rightarrow \bar{V} \otimes \Omega^1(\text{alg})$

$V, X$  compact  $\xrightarrow{\text{GAGA}}$  algebraic structure

VMHS<sup>p</sup>,  $\otimes, ( )^\vee$ , functorial pull-back  $f^*$

$\rightarrow$  functorial ring hom  $K_0(\text{VMHS}^p(X)) \xrightarrow{\text{MHC}^\mathbb{Q}} K^0(X)[\gamma^{\pm 1}]$   
alg. VB.

$[V] \longmapsto \sum_{i=0}^{\infty} [gr_F^i(V)] \cdot (\gamma)^i$

Note can forget algebraic struct. and map to topological K-theory

$$\rightarrow K_{top}^0(X)[\pm 1]$$

(3) Alg/ε

M. Saito: (a) MHM(X) abelian cat.  $\boxtimes$ , duality D

(b)  $D^b \text{MHM}(X) \xrightarrow{\text{rat.}} D_c^b(X)$  algebraic const. sheaves

$$\begin{array}{ccc} \cup & & \cup \\ \text{MHM}(X) & \xrightarrow[\text{faithful}]{\text{rat.}} & \text{Perv}(X) \end{array}$$

top: triangulated  
bottom: exact.

(c) Remark: weight filtration

$\mathcal{M}$  pure MHM  $\leadsto$   $\exists$  polarisation

$$\begin{array}{c} \text{choice} \\ \text{of pol.} \end{array} \text{rat}(\mathcal{M}) \xrightarrow{\sim} D(\text{rat}(\mathcal{M}))$$

(d) "Grothendieck calculus lifts"

$$f_!, f^!, f_*, f^*, \boxtimes, D.$$

Example:

$$Y \xleftarrow{i} X \xleftarrow{j} X \times Y$$

$$\text{triangle: } j_! j^* \rightarrow \text{id} \rightarrow i_* i^* \rightarrow$$

(e)  $X$  smooth; Def:  $\mathcal{M} \in \text{MHM}(X)$ ,  $\mathcal{M}$  smooth  $\Leftrightarrow$   
 $\dim_{\mathbb{C}} X = d$

$\text{rat}(\mathcal{M})$  is a local system, up to shift.

Saito  $\rightarrow$  there is an equiv. of categories

$$\begin{array}{ccc} \text{MHM}(X) & \xleftarrow{\text{smooth}} & \text{VMHS}^p(X) \\ \downarrow \mathbb{V}_H & & \downarrow \mathbb{V} \\ & \xleftarrow{\quad} & \end{array}$$

$$\mathbb{V}^H := \mathbb{V}_H[-d] \rightarrow \text{rat}(\mathbb{V}^H) = \mathcal{L}$$

Cor. (i)  $MHM(pt) \xleftarrow{\sim}$  mixed hodge structure  $MHS^p$

$$\mathbb{Q}_{pt}^H \longleftarrow \mathbb{Q} = \mathbb{Q}(0)$$

$$(ii) k: X \rightarrow pt \quad \text{rat} \left( H^i(k_{(1)}(X), k^* \mathbb{Q}_{pt}^H) \right) = H_{(0)}^i(X; \mathbb{Q}) \in MHS^p$$

(iii)  $U \subset X$  open, dense, smooth

$$IC_X^H := \text{Im} \left( H^0(j_* \mathbb{Q}^H[U]) \rightarrow H^0(j_* \mathbb{Q}^H[U]) \right)$$

$$\text{rat}(IC_X^H) = IC_X$$

$$(iv) k_0(V_{\text{an}}(X)) \longrightarrow k_0(MHM(X))$$

$$[Z \xrightarrow{f} K] \longmapsto [f_* \mathbb{Q}_Z^H]$$

(4) Saito:  $p \in \mathbb{Z}$  fixed.

Functor of  $\Delta$ -cd categories

$$gr_p^F DR: D^b(MHM(X)) \longrightarrow D_{\text{coh}}^b(X) \quad \begin{array}{l} \text{alg. cat} \\ \text{sheaves} \end{array}$$

- commutes with proper pushdown

$$gr_p^F DR(M) = 0, |p| \gg 0 \quad M \text{ fixed.}$$

$$X \text{ smooth and } X \xrightarrow[\text{open}]{i} \bar{X} \longleftarrow \bar{X} \setminus X = D \quad \begin{array}{l} \text{NCD} \\ \text{Normal crossing} \\ \text{divisor} \end{array}$$

$$\forall G \in VMHS^p(X) \rightarrow \text{log de Rham complex}$$

$$0 \rightarrow \bar{V} \xrightarrow{\bar{\nabla}} \bar{V} \otimes \Omega^1(\log D) \cdots \rightarrow \bar{V} \otimes \Omega^q(\log D) \rightarrow 0$$

filtered

$$\bar{F}_p \longrightarrow \bar{F}_{p+1} \otimes \Omega^1(\log D) \rightarrow \dots$$

$$(DR_{\log}(\bar{V}), \bar{F}^\bullet) = (\bar{V}, \bar{F}^\bullet) \otimes_{\mathbb{C}[X]} (\Omega^\bullet(\log D), P^\bullet)$$

$$\text{Saito: } gr_{\bar{F}}^F DR(j_* V^H) = gr_{\bar{F}}^F (DR_{\log}(\bar{V}))$$

$$MHM_X: \text{Kot } M$$

collected  
cat.

to shift.

$$MHC_* \mathcal{M} : K_0(MHM(X)) \longrightarrow G_0(X)[Y^{\pm 1}]$$

$$[\mathcal{M}] \longmapsto \sum_p [gr_p^F OR(\mathcal{M})] (-Y)^p$$

-commute with proper pushdown, ~~is~~

(i) " "  $\boxtimes$

(ii)  $X$  smooth

$$MHC_* (V^H) = MHC_*(V) \otimes \lambda_Y (\Omega_X^1)$$

Example:  $V = \mathbb{Q}_X$

$$\rightarrow MHC_*(X) = \lambda_Y (\Omega_X^1)$$

(ii)  $X = pt \rightarrow MHC_* = \mathbb{Z}_Y$