MSRI: Topology of Stratified Spaces 12th Sept. 2008

Jörg Schürmann: Characteristic classes of mixed Hodge modules

related work: Brasselet - Schürmann - Yokowa
Cappell - Libgober - Maxim - Shaneson

(1) Hodge genera

(2) Char. classes of VMHS : MHS^*

(3) Calculus of MHS

(4) Char classes of MHS : MHC_*

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(1) $MHS^{(p)}$ = abelian category of (graded pol.) $\mathbb{Q}$-mixed H.S.

— tensor product $\otimes$, duality $(\ )^\vee$

$(V, F^\cdot, W_\cdot)$ pure of weight $n \Leftrightarrow gr^W_k(V) = 0 \quad k \neq n$

pure weight n $\rightsquigarrow$ polarisation $\rightsquigarrow$ $V \xrightarrow{\sim} V^\vee(-n)$

Ring homom.
$$\begin{tikzcd} K_0(MHS^{(p)}) \arrow[r, "E"] \arrow[dr, "\text{\"Forget } W\text{\" } \chi_y"'] & \mathbb{Z}[u^{\pm 1}, v^{\pm 1}] \arrow[d, "u=-y,\ v=1"] \\ & \mathbb{Z}[y^{\pm 1}] \end{tikzcd}$$

<u>Ex.</u> (Deligne)

X alg. $/\mathbb{C}$ $H^{\cdot}_{(c)}(X; \mathbb{Q}) \in MHS^p$

$$Y \underset{cl}{\hookrightarrow} X \underset{open}{\hookleftarrow} X \setminus Y$$

$$\dots \to H^i_c(U) \to H^i_c(X) \to H^i_c(Y) \to \dots$$

$\rightsquigarrow$ ring hom. $K_0(Var/pt) \longrightarrow K_0(MHS^p)$
$$[X] \longmapsto [H^*_c(X)]$$

Question: X alg., compact, smooth.
Is there a char. class of vector bundles
$$E(X) = E([H^*_c(X)]) \overset{?}{=} \deg\ \underbrace{(cl^*(TM) \cap [M])}_{\text{top ch \#}}$$

Answer: No!

X, X' elliptic curves $\quad X' \xrightarrow{d:1} X$ cover

top # $\Rightarrow E(X') = d\,E(X)$ but $E(X) = (1-u)(1-v) = E(X') \neq 0$

Yes for χ_y - Hirzebruch '54

(2) X smooth alg. $/\mathbb{C}$

VMHS = abelian cat. of var of $\mathbb{Q}$-MHS

- $\otimes$ prod, duality $(\ )^\vee$.

$(L, F^\cdot, W_\cdot)$ $\quad L$ loc. syst. on X, $W_\cdot$ Filtr. on L

$V = L \otimes_{\mathbb{C}_X} \mathcal{O}_X$ holomorphic VB with $\nabla = 1 \otimes d$

$\nabla \circ \nabla = 0$, $L = \ker(\nabla)$

$\nabla : V \to V \otimes \Omega^1_X$

$\overset{\cup}{F^p} \xrightarrow{\nabla} F^{p-1} \otimes \Omega^1_X$

Rem: V pure + choice of polarisation $\Rightarrow L \xrightarrow{\sim} L^\vee$

Poincaré local system.

$VMHS^p$ $\quad$ p-polarisation — quasi $\overset{\text{unipotent}}{\vee}$ monodromy at ∞ / admissable at ∞ / graded pol.

$X \underset{\text{open}}{\subset} \bar{X} \supset D = \bar{X} \setminus X$ $\quad$ DNC

Deligne Ext : $(\bar{V}, \bar{F}^\cdot, \bar{\nabla})$ $\quad$ ~~$\bar{\nabla}$~~ $\bar{\nabla} : \bar{V} \to \bar{V} \otimes \Omega^1(\log D)$

$V, \bar{X}$ compact $\xrightarrow{\text{Gaga}}$ algebraic structure

$VMHS^p$, $\otimes$, $(\ )^\vee$, functorial pull-back f^*

$\Rightarrow$ functorial ring hom $\quad K_0(VMHS^p(X)) \xrightarrow{MHC^y} K^0(X)[y^{\pm 1}]$ alg. VB.

$[V] \longmapsto \sum_p [gr^p_F(V)](-y)^p$

Note can forget algebraic struct. and map to topological K-theory

$\longrightarrow K^0_{top}(X)[y^{\pm 1}]$

(3) Alg/$\mathbb{C}$

M. Saito: (a) $MHM(X)$ abelian cat., $\boxtimes$, duality D

(b)
$$\begin{array}{ccc} D^b MHM(X) & \xrightarrow{rat.} & D^b_c(X) \text{ algebr. constr. sheaves} \\ \cup & & \cup \\ MHM(X) & \xrightarrow[\text{faithful}]{rat} & Perv(X) \end{array}$$

top: triangulated
bottom: exact.

(c) Remark: weight filtration

M pure MHM $\rightsquigarrow \exists$ polarisation

$\xrightarrow[\text{of pol.}]{\text{choice}} rat(M) \xrightarrow{\sim} D(rat(M))$

(d) "Grothendieck calculus lifts"

$f_!, f^!, f_*, f^*, \boxtimes, D.$

<u>Example</u>: $Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X \setminus Y$

triangle: $j_! j^* \to id \to i_* i^* \to$

(e) X smooth; $\dim_{\mathbb{C}} X = d$ Def: $M \in MHM(X)$, M smooth $\Leftrightarrow$ $rat(M)$ is a local system, up to shift.

$\xrightarrow{\text{Saito}}$ there is an equiv. of categories

$$\begin{array}{ccc} MHM(X)_{smooth} & \longleftarrow & VMHS^p(X) \\ V_H & \longmapsfrom & V \end{array}$$

$V^H := V_H[-d] \rightsquigarrow rat(V^H) = L$

isolated
cct.

to shift.

<u>Cor.</u> (i) $MHM(pt) \xleftarrow{\sim}$ mixed hodge structure MHS^p

$$\mathbb{Q}^H_{pt} \longleftarrow \mathbb{Q} = \mathbb{Q}(0)$$

(ii) $k : X \to pt$ $\quad rat\left(H^i\left(k_{(!)(*)}\, k^* \mathbb{Q}^H_{pt}\right)\right) = H^i_{(c)}(X;\mathbb{Q}) \in MHS^p$

(iii) $\mathcal{U} \subset X$ open, dense, smooth

$$IC^H_X := \mathrm{Im}\left(H^0(j_!\, \mathbb{Q}^H[d]) \to H^0(j_*\, \mathbb{Q}^H[d])\right)$$

$$rat(IC^H_X) = IC_X$$

(iv) $K_0(Var/X) \longrightarrow K_0(MHM(X))$

$$[Z \xrightarrow{f} X] \longmapsto [f_! \mathbb{Q}^H_Z]$$

(4) <u>Saito:</u> $p \in \mathbb{Z}$ fixed.

Functor of Δ-ed categories

$$gr^F_p DR : D^b MHM(X) \longrightarrow D^b_{coh}(X) \quad \text{alg. coh sheaves}$$

- commutes with proper pushdown

$$gr^F_p DR(\mathcal{M}) = 0, \quad |p| \gg 0 \quad \mathcal{M} \text{ fixed.}$$

X smooth and $X \xrightarrow[\text{open}]{j} \bar{X} \hookleftarrow \bar{X} \setminus X = D$ NCD Normal crossing divisor

$V \in VMHS^p(X) \to$ log deRahm complex

$$0 \to \bar{V} \xrightarrow{\bar{\nabla}} \bar{V} \otimes \Omega^1(\log D) \cdots \to \bar{V} \otimes \Omega^d(\log D) \to 0$$

filtered

$$\bar{F}_p \longrightarrow \bar{F}_{p-1} \otimes \Omega^1(\log D) \to \cdots$$

$$(DR_{\log}(\bar{V}), F^\cdot) = (\bar{V}, \bar{F}) \otimes_{\mathcal{O}_X} (\Omega^\cdot(\log D), P^\cdot)$$

Saito: $gr^F_{\cdot}\, DR(j_* V^H) = gr^{\bullet}_{\bar{F}}\,(DR_{\log}(\bar{V}))$

~~MHC_X : Kot M~~

$$MHC_* \cancel{(X)} : K_0(MHM(X)) \longrightarrow G_0(X)[y^{\pm 1}]$$

$$[M] \longmapsto \sum_p [gr^F_{-p} DR(M)](-y)^p$$

- commute with proper pushdown, $\boxtimes$,

(i) " " $\boxtimes$

(ii) X smooth

$$MHC_*(V^H) = MHC^*(V) \otimes \lambda_y(\Omega^1_X)$$

<u>Example</u>: $V = \mathbb{Q}_X$

$$\longrightarrow MHC_*(X) = \lambda_y(\Omega^1_X)$$

(iii) $X = pt \rightarrow MHC_* = \chi_y$