Sobolev imbeddings: a phase-space probabilistic view point

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Paley-Zygmund

Take a sequence $(a_n) \in l^2(\mathbb{Z})$.
Let $\varepsilon_n(\omega) = \pm 1$ be random independent Bernoulli variables.
Then almost surely in ω, the function on the circle $M = \mathbb{R}/\mathbb{Z}$

$$f_\omega(x) = \sum_{n \in \mathbb{Z}} \varepsilon_n(\omega) a_n e^{2i\pi nx}$$

belongs to $L^p(M)$ for any $p < \infty$.

The rules of the game

Let (M, g) be a compact Riemannian manifold of dimension d, and X a space of functions on M. One wants to make sense to the affirmation

" For almost all function in X, this property holds true "

Unfortunately, we know that there is no probability on the unit sphere of the energy space $X = L^2(M)$ which is invariant by the group of isometries of $L^2(M)$.
In this talk we will show that there is a (in fact many...) natural probability P on X such that roughly

- 1. Functions at very different scales are independent.
- 2. At each scale, the probability P is a model for a uniform repartition of the energy in the phase space, with in some sense, the Liouville measure as semi-classical limit.

Outline

1. The probability on Besov spaces
2. The Sobolev “almost sure” inequality
3. Some simple applications to waves
   - The linear wave equation on $\mathbb{R}_t \times M$
   - Non-linear supercritical waves
   - The damped wave equation

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The uniform probability on functions at scale h

Let $I = [c, c'], 0 \leq c < c' < \infty$ be a given interval in $[0, \infty[$. Let $(e_k)_{k\geq 0}$ be an orthonormal basis of eigenfunctions of $-\triangle$ on (M, g), $-\triangle(e_k) = \omega_k^2 e_k$. Let $E_{I,h}$ be the finite dimensional subspace of $L^2(M)$

$$E_{I,h} = \{u = \sum_{k\in I_h} a_k e_k(x),\ a_k \in \mathbb{C}\}, \quad I_h = \{k,\ h\omega_k \in I\} \tag{2.1}$$

Set $N(I, h) = dim(E_{I,h})$. Then by Weyl formula one has for all $h \in]0, 1]$

$$N(I, h) = (2\pi h)^{-d}\mathrm{Vol}(M)\mathrm{Vol}(S^{d-1}) \int_I \rho^{d-1} d\rho + O(h^{-d+1}) \tag{2.2}$$

Let $S_{I,h}$ be the unit sphere of the Euclidian space $E_{I,h} = \mathbb{C}^{N(I,h)}$.

Definition

We denote by $P_{I,h} = P$ the uniform probability on the sphere $S_{I,h}$.

The law of the value at a given $x \in M$

For all $x \in M$, let $b_x = (e_k(x))_{k \in I_h} \in \mathbb{C}^{N(I,h)}$ and let ev_x be the random variable on $(S_{I,h}, P_{I,h})$

$$ev_x(u) = u(x) = \sum_{k \in I_h} a_k e_k(x) = (a|b_x) = (a|\frac{b_x}{|b_x|})|b_x| \qquad (2.3)$$

The vector $\varepsilon_x = \frac{b_x}{|b_x|}$ has norm 1 . Let $\Phi(t) = P(|(a|\varepsilon)| > t)$ where ε is an arbitrary unit vector. The following lemma is essentially the so called "Poincare lemma" which is not of Poincare, but is a lemma of Borel...

Lemma

$$\Phi(t) = \mathbf{1}_{t \in [0,1[}(1 - t^2)^{N(I,h)-1} \qquad (2.4)$$

and one has for all $r \geq 0$

$$P\Big(|ev_x| > r\Big) = P\Big(|(a|\varepsilon_x)| > r/|b_x|\Big) = \Phi(r/|b_x|) \qquad (2.5)$$

The law of the value f(x)

The following lemma is due to Hörmander (1968)

Lemma

There exists $C > 0$ such that for all $x \in M$ and all $h \in]0,1]$ one has

$$||b_x|^2/N(I,h) - 1/Vol(M)| \leq Ch \tag{2.6}$$

From this lemma, we see that the law $L_{I,h;x}$ of the complex value random variable ev_x is for h small and uniformly in $x \in M$ close to the standard Gaussian law $N(0,\sigma) = (\pi\sigma^2)^{-1}e^{-|z|^2/\sigma^2}$, with $\sigma^2 = (\mathrm{Vol}(M))^{-1}$, since $L_{I,h;x}$ is clearly invariant by rotation, one has,
$\int_{|z|>r}(\pi\sigma^2)^{-1}e^{-|z|^2/\sigma^2}L(dz) = e^{-r^2/\sigma^2}$, and

$$\int_{|z|>r} dL_{I,h;x} = P\Big(|ev_x| > r\Big) = \mathbf{1}_{r<|b_x|}(1-(r/|b_x|)^2)^{N(I,h)-1} \simeq_{h\to 0} e^{-r^2\mathrm{Vol}(M)} \tag{2.7}$$

Correlations

Let x, y two points in M. One wants to study the joint law of the random variables ev_x et ev_y. This is closely linked with Weyl asymptotic on the spectral projector

$$\Pi_{I,h}(x,y) = \sum_{k \in I_h} e_k(x) e_k(y) \tag{2.8}$$

One will see that :

The metric $d_g(x,y)$ can be reconstructed by the correlation between the random variables ev_x et ev_y.

There is also results by Zelditch on "smooth approximations" of the spectral projector, like

$$\sum_k \varphi(\omega_k - h^{-1}) e_k(x) e_k(y) \tag{2.9}$$

where φ is smooth with compact support. Notice that in 2.9, we do not have any smoothing at the end points of the interval I_h.

The model $M = \mathbb{R}/2\pi\mathbb{Z}$

In this flat case, we set $e_k(x) = (2\pi)^{-1/2} e^{ikx}$ for $k \in \mathbb{Z}$. One has with $I = [0, \Lambda]$, and N the greatest integer such that $hN \leq \Lambda$

$$\Pi_{I,h}(x,y) = (2\pi)^{-1} \sum_{-N}^{+N} e^{ik(x-y)} = (2\pi)^{-1} \frac{\sin((N+1/2)|x-y|)}{\sin|\frac{x-y}{2}|} \quad (2.10)$$

In particular, one has $\mathrm{Vol}(M) = 2\pi$, $N(I,h) = 2N+1$, and

$$\begin{aligned} |b_x|^2 &= (2\pi)^{-1}(2N+1) \\ (\varepsilon_x|\varepsilon_y) &= \frac{1}{2N+1} \frac{\sin((N+1/2)|x-y|)}{\sin|\frac{x-y}{2}|} \end{aligned} \quad (2.11)$$

One sees that formula (2.6) on the asymptotic of $|b_x|^2$ is trivially exact (equirepartition of the eigenfunctions). Also, the correlation factor $(\varepsilon_x|\varepsilon_y)$ is $O(h)$ for $x \neq y$ and is oscillating

Correlation: where one recovers the metric

In the flat case, for $y = x + hz$, one gets

$$\lim_{h \to 0} (\varepsilon_x | \varepsilon_{x+hz}) = \frac{\sin(\Lambda z)}{\Lambda z} \tag{2.12}$$

For a general (M, g), one has the following result. In normal geodesics coordinates centered x, one has for all $R > 0$ uniformly in $z \in \mathbb{R}^d, |z| \leq R$

$$\lim_{h \to 0} (\varepsilon_x | \varepsilon_{x+hz}) = \frac{\int_{c \leq |\xi| \leq c'} e^{iz\xi} d\xi}{\int_{c \leq |\xi| \leq c'} d\xi} \tag{2.13}$$

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The basic estimate

Theorem

There exists $c_1 > 0$ and for all $c_2 < Vol(M)$ a constant C_0 such that for all $h \in]0,1]$ and all $\Lambda \geq 1$ one has

$$P_{h,2}\big(g \in S_{h,2},\ \|g\|_{L^\infty} > \Lambda\big) \leq C_0 h^{-c_1} e^{-c_2 \Lambda^2} \qquad (3.1)$$

The proof gives $c_1 = d(1 + d/2)$....

Sobolev almost sure

Let X be the product space

$$X = \Pi_{n=0}^{\infty} S_{2^{-n},2} \tag{3.2}$$

We equip X with the probability $P = \Pi_{n=0}^{\infty} P_{2^{-n},2}$ (independence of the different scales). Let j_σ be the map from X in the Besov space $B^{\sigma}_{2,\infty}$

$$\begin{aligned} X &\to B^{\sigma}_{2,\infty} \\ g = (g_n)_{n\geq 0} &\mapsto j_\sigma(g) = \sum_{n=0}^{\infty} 2^{-n\sigma} g_n \end{aligned} \tag{3.3}$$

As an immediate corollary of the theorem (3.1) we get

Corollary

For any $\sigma > 0$, one has $P(j_\sigma(g) \in L^\infty) = 1$.

Sobolev almost sure

Remark

What corollary (3.2) says, is that one has a almost sure injection of $B^{\sigma}_{2,\infty}$ into L^{∞} for all $\sigma > 0$, thus a gain of $d/2$ derivatives in comparison of the classical Sobolev embedding.

There is a hope to use this to get "almost sure results" in situations like supercritical non linear equations, where we know that it is too much difficult to get any results true for all data. This was used with different probabilities in recent works by N. Burq and N. Tzvetkhov.

The meaning of corollary (3.2) is the following:
If one has a family g_h of functions at scale h and of energy 1, and if we assume that the distribution of their energy is random in the phase space with respect to the Liouville measure, then these functions are almost bounded, in the sense that $sup_h h^s \|g_h\|_{L^\infty}$ is finite for any $s > 0$. Liouville is intrinsic on the cotangent space T^*M, but the function $|\xi|_x$ on T^*M (the metric), both defines the scale, and the L^2 space (energy).

Proof of the corollary

Let $m_n = Ae^{n\sigma/2}$ et B_n the subset of $S_{2^{-n},2}$

$$B_n = \{g_n,\ \|g_n\|_{L^\infty} \le m_n\}$$

By (3.1) one has

$$P_{2^{-n},2}(B_n) \ge 1 - C_0 2^{nc_1} e^{-c_2 A^2 e^{n\sigma}} \tag{3.4}$$

Let B be the subset of X, $B = \Pi_{n=1}^{\infty} B_n$. For $g = (g_n) \in B$ and $f = j_\sigma(g)$, one has

$$\|f\|_{L^\infty} \le \sum_{n=0}^{\infty} 2^{-n\sigma} \|g_n\|_{L^\infty} \le \sum_{n=0}^{\infty} 2^{-n\sigma} m_n = \frac{A}{1 - e^{-\sigma/2}} \tag{3.5}$$

Thus for all $f \in j_\sigma(B)$, $f \in L^\infty$, using (3.5) and (3.4)

$$P(B) = \Pi_{n=0}^{\infty} P_{2^{-n},2}(B_n) \ge \Pi_{n=0}^{\infty}\left(1 - C_0 2^{nc_1} e^{-c_2 A^2 e^{n\sigma}}\right) \ge 1 - \varepsilon$$

with $\varepsilon > 0$ small if A is large enough .

Proof of theorem 3.1

There exists $0 < C_1 < C_2$ such that for all $h \in]0,1]$ and $u \in S_{I,h}$ one has $C_2 \leq \|u\|_{L^\infty} \leq C_1 h^{-d/2}$, and $\|\nabla_x u\|_{L^\infty} \leq C_1 h^{-d/2-1}$, thus we get

$$sup_x \ |u(x)| \leq sup_{\alpha \in A} \ |u(x_\alpha)| + \varepsilon\Lambda \tag{3.6}$$

as soon as $x_\alpha, \alpha \in A$ is a grid of points in M with typical mesh size $\varepsilon\Lambda h^{1+d/2}/C_1$. Let $\beta \in]0,1[$. By (2.5), (2.4), lemma (2.3), and $(1-t^2)^N \leq e^{-Nt^2}$ for $t \in [0,1]$, there exists $h_0 > 0$ such that for $h \in]0, h_0]$ one has

$$\begin{aligned} P\Big(u \in S_h, \ \|u\|_{L^\infty} > \Lambda\Big) &\leq \sum_{\alpha \in A} P\Big(|ev_{x_\alpha}| > (1-\varepsilon)\Lambda\Big) \\ &\leq \sum_{\alpha \in A} \mathbf{1}_{(1-\varepsilon)\Lambda \leq |b_{x_\alpha}|} (1 - (1-\varepsilon)^2\Lambda^2/|b_{x_\alpha}|^2)^{N(I,h)-1} \\ &\leq \sum_{\alpha \in A} e^{-(N(I,h)-1)(1-\varepsilon)^2\Lambda^2/|b_{x_\alpha}|^2} \leq card(A) e^{-\beta Vol(M)(1-\varepsilon)^2\Lambda^2} \end{aligned} \tag{3.7}$$

Since $card(A) \leq C\varepsilon^{-d}\Lambda^{-d}h^{-d(1+d/2)}$, we get our result.

Outline

various choices of probabilities

Let $E_k = E_{I,2^{-k}}$ be the Littlewood-Paley blocks. If $(r, u), r \geq 0, \|u\| = 1$ are polar coordinates on E_k, we may equipped the Euclidian space E_k with a probability

$$P_k = \mu_k(dr)\underline{P}(du)$$

where $\underline{P}(du)$ is uniform on the sphere, and $\mu_k(dr)$ is a probability on $[0, \infty[$. Then on the product space $\Pi_k E_k$, one gets a probability

$$P = \Pi_k P_k$$

for which different scales are independent. The choice of the measure $\mu_k(dr)$ is at our disposal.

Let M be a compact Riemannian manifold, $X = B^1_{2,\infty} \times B^0_{2,\infty}$ and consider the linear wave equation

$$(\partial_t^2 - \Delta)u = 0, \quad (u\,|_{t=0}, \partial_t u\,|_{t=0}) = (u_0, u_1) \in X \tag{4.1}$$

Let P be the natural probability on the unit ball B_X of X (independence of scales, and uniform on the unit ball of each Littlewood-Palay block).

Theorem

For any $\varepsilon > 0$, on has

$$P(\{(u_0, u_1) \in B_X,\ u \in W^{1-\varepsilon,\infty}([0,\infty[\times M)\}) = 1 \tag{4.2}$$

Let M be a compact Riemannian manifold of dim 3 , and let p be an odd integer.

$$(\partial_t^2 - \Delta)u + u^p = 0, \quad u\mid_{t=0} = u_0 \in H^1(M), \partial_t u \mid_{t=0} = u_1 \in L^2(M) \quad (4.3)$$

The existence, for any data $(u_0, u_1) \in H^1(M) \cap L^{p+1}(M) \times L^2(M)$, of weak solutions defined for $t \in [0, \infty[$ for this non linear wave equation is known (W.Strauss and J.L. Lions). These weak solutions satisfies

$$E(u)(t) = \frac{1}{2} \int_M |\nabla u|^2 + |\partial_t u|^2 + \int_M \frac{|u|^{p+1}}{p+1} \leq E(u)(0). \quad (4.4)$$

but nothing is known on their uniqueness in the super-critical case $p \geq 7$ in dim 3.

In the sub-critical $p < 5$ or critical case $p = 5$, uniqueness and continuous dependance on the data is known (works of Ginibre-Velo and Shatah-Struwe).

In the super-critical case, there are strong instabilities results for the Cauchy problem near $t = 0$ (works of Lebeau and Coliander-Keel-Tao).

Let $p < +\infty$ be an odd integer.

Theorem

With probability 1 *on the data* $(u_0, u_1) \in B_X$, *one has* $u_0 \in L^{p+1}(M)$, *and there exists* $T = T_{(u_0,u_1)} > 0$ *and a strong solution* u *of 4.3 in the space*

$$C^0((0,T); B^1_{2,\infty}) \cap C^1((0,T); B^0_{2,\infty} \cap L^2((0,T); W^{1-\epsilon,\infty}(M)).$$

The proof is simple: if u_l is the solution of the linear wave equation with the same Cauchy data , one has with probability 1, $u_l \in W^{1-\varepsilon,\infty}$, and we search the solution u of the non-linear equation on the form $u = u_l + v$ where v is solution of

$$(\partial_t^2 - \Delta)v = (u_l + v)^p, \qquad v\mid_{t=0} = \partial_t v\mid_{t=0} = 0 \tag{4.5}$$

For $T > 0$ small enough, this admits a unique fix point in the space

$$C^0(I; H^{2-\epsilon}(M) \cap C^1(I; L^2(M)) \cap L^p(I; W^{1-\epsilon,q}(M)) \cap W^{1-\epsilon,p}(I; L^q(M))$$

with $I = (0, T)$, where the Strichartz (p, q) exponents are close to $(2, \infty)$. There is also a uniqueness result with probability 1.

a conjecture

Conjecture With probability 1 on the data, the non-linear equation 4.3 admits a unique global solution u which belongs to the space $W^{1-\varepsilon,\infty}$.

Of course, part of the difficulty is to define properly the probability, and to verify that there is no spurious interactions between the different scales with probability 1.

Even if true, notice that the exceptional set may be not small in the Baire sens

Let M be a compact Riemannian manifold of dim d, and let $a(x)$ be a smooth non negative function on M. The damped wave equation is

$$(\partial_t^2 - \Delta)u + 2a(x)\partial_t u = 0, \quad u\mid_{t=0} = u_0 \in H^1(M), \partial_t u\mid_{t=0} = u_1 \in L^2(M) \tag{4.6}$$

The spectrum Σ is the set of $\lambda = i\omega - \sigma$ such that the equation 4.6 admits a non trivial solution $u(t,x) = e^{t\lambda}v(x)$, and the generalized eigenspaces $\mathcal{E}_\lambda \subset H^1 \oplus L^2 = \mathcal{H}$ are finite dimensional.
As before, we set for a given interval I of $\mathbb{R}$

$$E_{I,h} = \{u = \sum_{\lambda \in I_h} f_\lambda(x),\ f_\lambda \in \mathcal{E}_\lambda\}, \quad I_h = \{\lambda \in \Sigma,\ hIm(\lambda) \in I\} \tag{4.7}$$

One has a Weyl formula for $N(I,h) = dim(E_{I,h}) \simeq Cteh^{-d}$ and we equipped $E_{I,h}$ with the Euclidian structure of $H^1 \oplus L^2 = \mathcal{H}$, and we put the uniform probability on the unit ball of $E_{I,h}$.

For $\rho \in S^*M$, and $t \geq 0$, let $\phi(t,\rho)$ be the geodesic flow and set

$$\underline{a}(t,\rho) = \frac{1}{t}\int_0^t a(x(\phi(s,\rho)))ds$$

Then the Birkhoff function $Bir(\rho)$ is defined Liouville almost everywhere as

$$Bir(\rho) = \lim_{t\to\pm\infty} \underline{a}(t,\rho) \in [\min a, \max a]$$

We set $\beta_- = \inf ess\ Bir(\rho)$, $\beta_+ = \sup ess\ Bir(\rho)$. One has $\min a \leq \beta_- \leq \beta_+ \leq \sup a$ and strict inequalities may occur. In case of ergodicity, one has $\beta_- = \beta_+ = \oint_M a$. Set

$$\mu_R(I) = \frac{\sharp\{\lambda = i\omega - \sigma \in \Sigma,\ \sigma \in I,\ |\omega| \leq R\}}{\sharp\{\lambda = i\omega - \sigma \in \Sigma,\ |\omega| \leq R\}}$$

Then the support of any weak limit of the sequence of probabilities μ_R is contained in the interval $[\beta_-, \beta_+]$ (Sjostrand). There is also examples (Lebeau, complex deformation), where there exists $c > 0$ such that

$$\sigma \in [\beta_- + c, \beta_+ - c], \forall \lambda = i\omega - \sigma \in \Sigma$$

One may have exceptionnal sequences $\lambda_n = i\omega_n - \sigma_n \in \Sigma$ such that $\lim \sigma_n < \beta_-$. However, with

$$C(t) = \inf_{\rho \in S^* M} \underline{a}(t, \rho)$$

the limit $\lim_{t\to+\infty} C(t) = C(\infty)$ exists, and one has $\lim \sigma_n \geq C(\infty)$. Let α be the best constant in exponential decay of the energy

$$\alpha = \sup\{c \geq 0,\ \exists B\ \forall u \in \mathcal{H},\ E(u,t) \leq Be^{-ct} E(u,0)\}$$

Theorem

$$\alpha = 2 \min\{C(\infty), \min_{\lambda = i\omega - \sigma \in \Sigma} \sigma, \} \tag{4.8}$$

What can be said on the decay of the energy in a probabilistic setting on the data ?

Let us assume that one has $\beta_- > 0$.

Theorem

Let $\alpha_p = 2\min\{\beta_-, \min_{\lambda=i\omega-\sigma\in\Sigma}\sigma, \}$ *. For all* $\varepsilon, \nu > 0$*, there exists a set* E *with* $P(E) \geq 1-\nu$*, such that for any* $(u_0, u_1) \in E$*, there exists* B *such that*

$$E(u,t) \leq Be^{-(\alpha_p-\varepsilon)t}E(u,0) \tag{4.9}$$

Theorem

For all $\varepsilon, \nu > 0$*, there exists* $T > 0$ *such that*

$$P(E(T,u_0) > \varepsilon, \quad E(0,u_0) = 1) \leq \nu \tag{4.10}$$