Inverse hyperbolic problems and optical (acoustic) black holes.

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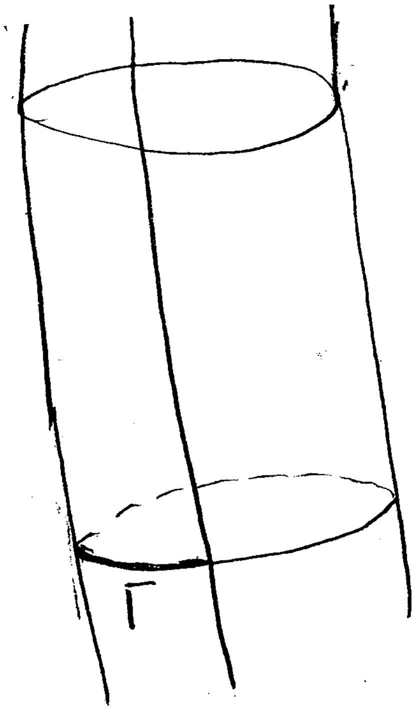
1. Initial – boundary value problem



Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be an open subset of $\partial\Omega$.

Consider a hyperbolic equation in the cylinder $\Omega\times\mathbb{R}$:

$$(1) \qquad \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_j}\left(\sqrt{(-1)^n g(x)}\, g^{jk}(x) \frac{\partial u(x_0,x)}{\partial x_k}\right) = 0,$$

where $x=(x_1,\dots,x_n)\in\overline{\Omega}$, $x_0\in\mathbb{R}$ is the time variable, the coefficients in (1) are independent of x_0.

$[g_{jk}(x)]_{j,k=0}^{n} = \left([g^{jk}(x)]_{j,k=0}^{n}\right)^{-1}$ is a pseudo-Riemannian metric with the Minkowsky signature, i.e. the quadratic form $\sum_{j,k=0}^{n} g^{jk}(x)\xi_j\xi_k$ has the signature $(1,-1,-1,\dots,-1)$ for all $x\in\overline{\Omega}$,

$g(x) = \det[g_{jk}(x)]_{j,k=0}^{n}$.

Note that $(-1)^n g(x) > 0$

We assume that

(2) $g^{00}(x) > 0$, $x \in \overline{\Omega}$, i.e. $(1, 0, \dots, 0)$ is not a characteristic direction,

(3) $(0, \xi_1, \dots, \xi_n)$ is not a characteristic direction for any $(\xi_1, \dots, \xi_n) \neq (0, \dots, 0)$ and any $x \in \Omega$

Note that (3) is equivalent to the condition that

(3') $g_{00}(x) > 0$, $x \in \overline{\Omega}$,

i.e. that $(1, 0, \dots, 0)$ is a time-like direction.

We consider the initial-boundary value problem for the equation (1) in the cylinder $\Omega \times \mathbb{R}$

(4) $u(x_0, x) = 0$ for $x \in \Omega$, $x_0 \ll 0$,

(5) $u(x_0, x)\big|_{\partial\Omega \times \mathbb{R}} = f(x_0, x')$, $x' \in \partial\Omega$,

where $f(x_0, x')$ has a compact support in $\partial\Omega \times \mathbb{R}$

Let Λf be the Dirichlet-to-Neumann (DN) operator, i.e.

(6) $$\Lambda f = \sum_{j,k=0}^{n} g^{jk}(x) \frac{\partial u(x_0, x)}{\partial x_j} \nu_k(x) \Big(- \sum_{p,r=1}^{n} g^{pr}(x) \nu_p \nu_r \Big)^{-\frac{1}{2}} \Big|_{\partial\Omega \times \mathbb{R}}$$

where $u(x_0, x)$ is the solution of (1), (4), (5)

$\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the unit outward normal vector to $\partial\Omega$.

Consider a change of variables of the form:

$$\hat{x}_0 = x_0 + a(x),$$
(7)
$$\hat{x} = \varphi(x),$$

where $\varphi(x)$ is a diffeomorphism of $\overline{\Omega}$ onto some domain $\hat{\overline{\Omega}}$ such that $\overline{\Gamma} \subset \partial\hat{\Omega}$, $\varphi(x) = x$ on $\overline{\Gamma}$, $a(x) = 0$ on $\overline{\Gamma}$. Note that (7) is an identity map on $\overline{\Gamma} \times \mathbb{R}$.

Note that the map (7) transform (1) into an equation of the same form in $\hat{\Omega} \times \mathbb{R}$.

The following theorem holds:

<u>Theorem 1</u> Let L and $\hat{L}$ be two operators of the form (1) in $\Omega \times \mathbb{R}$ and $\hat{\Omega} \times \mathbb{R}$ respectively. Consider initial-boundary value problems of the form (4), (5) for L and $\hat{L}$. Suppose $\Lambda f = \hat{\Lambda} f$ on $\Gamma \times \mathbb{R}$ for all $f \in C_0^\infty(\Gamma \times \mathbb{R})$ where $\Lambda, \hat{\Lambda}$ are DN operators for $L, \hat{L}$ respectively. Suppose that conditions (2) and (3) hold for L and $\hat{L}$. Then there exists a map of the form (7) such that

$$(8)\quad [\hat{g}^{jk}(\hat{x})]_{j,k=0}^{n} = J^T(x)\,[g^{jk}(x)]_{j,k=0}^{n}\,J(x),$$

where $\left([\hat{g}^{jk}(\hat{x})]_{j,k=0}^{n}\right)^{-1}$ is the metric tensor for $\hat{L}$ and $J(x)$ is the Jacobi matrix of (7)

Remark 1 It is enough to know the DN operator on $\Gamma \times (0, T_0)$ for some $T_0 > 0$ instead of $\Gamma \times \mathbb{R}$. More precisely let T_+ be the smallest number such that $\mathcal{D}_+(\overline{\Gamma} \times \{x_0 = 0\}) \supset \overline{\Omega} \times \{x_0 = T_+\}$ where $\mathcal{D}_+(\overline{\Gamma} \times \{x_0 = 0\})$ is the forward domain of influence of $\overline{\Gamma} \times \{x_0 = 0\}$ corresponding to (1). Analogously let T_- be the smallest number such that

$$\mathcal{D}_-(\overline{\Gamma} \times \{x_0 = T_-\}) \supset \overline{\Omega} \times \{x_0 = 0\}$$ where $\mathcal{D}_-(\overline{\Gamma} \times \{x_0 = T_-\})$ is the backward domain of influence of $\overline{\Gamma} \times \{x_0 = T_-\}$

If $T_0 > T_- + T_+$ then $\Lambda = \hat{\Lambda}$ on $\Gamma \times (0, T_0)$ implies (8) i.e. the isometry of metrics $[g_{jk}(x)]$ and $[\hat{g}_{jk}(\hat{x})]$

The equation of the propagation of light in the moving dielectric medium

It was discovered by Gordon (1923) that the equation of the propagation of light in a moving medium is given by the hyperbolic equation of the form (1) with the metric tensor

(9) $$g^{jk}(x) = \eta^{jk} + (n^2(x) - 1)\, v^j(x)\, v^k(x),$$

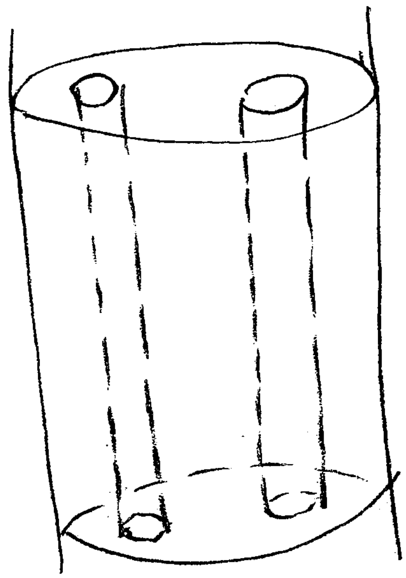
$$0 \le j, k \le n \quad , \quad n = 3,$$

where $[\eta^{jk}]$ is the Lorentz metric tensor:
$\eta^{jk} = 0$ when $j \ne k$, $\eta^{00} = 1$, $\eta^{jj} = -1$, $1 \le j \le n$,
$x_0 = t$ is the time, $n(x) = \sqrt{\varepsilon(x)\mu(x)}$ is the refraction index,
$w(x) = (w_1(x), w_2(x), w_3(x))$ is the velocity of the flow,

$$v^0(x) = \left(1 - \frac{|w(x)|^2}{c^2}\right)^{-\frac{1}{2}} \quad , \quad v^j(x) = \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}} \frac{w_j(x)}{c},$$

$1 \le j \le 3$,
$(v^{(0)}, v^{(1)}, v^{(2)}, v^{(3)})$ is the four-velocity field of the flow,
c is the speed of light in the vacuum.
We shall call the equation (1) with metric (9) the Gordon equation

Inverse problem for the Gordon equation



Let Ω be a smooth domain in $\mathbb{R}^n$ of the form

$$\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega_j},$$

where Ω_0 is simply-connected smooth domains Ω_j, $1 \le j \le m$, are smooth called obstacles, $\overline{\Omega_j} \subset \Omega_0$, $1 \le j \le m$, $\overline{\Omega_j} \cap \overline{\Omega_k} = \emptyset$ when $j \ne k$.

We shall consider the following initial-boundary value problem for the Gordon equation:

$$u(x_0, x) = 0 \quad \text{for } x_0 \ll 0, \ x \in \Omega,$$

(10) $$u\big|_{\partial\Omega_j \times \mathbb{R}} = 0, \ 1 \le j \le m,$$

$$u\big|_{\partial\Omega_0 \times \mathbb{R}} = f(x_0, x'),$$

i.e. $\partial\Omega_0 = \Gamma$ in the notations of Theorem 1.

Note that

$$g^{00}(x) = 1 + (n^2-1)(v^0(x))^2 > 0,$$

i.e. the direction $(1, 0, \dots, 0)$ is not characteristic

The condition that any direction $(0, \xi_1, \dots, \xi_n)$ is not characteristic holds when

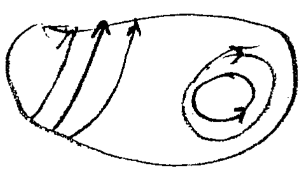
(11) $$|w(x)|^2 < \frac{c^2}{n^2(x)}$$

We shall impose some restrictions on the flow $w(x)$.

Let $x = x(s)$ be a trajectory of the flow, i.e.

$$\frac{dx}{ds} = w(x(s)), \quad 0 \le s \le 1,$$

where $w(x(s)) \neq 0$ for $0 \le s \le 1$. Consider the trajectories that start and end on $\partial\Omega_0$ or are close curves in Ω

2)  We assume that such trajectories are dense in $\overline{\Omega}$.

$\partial\Omega_0$

<u>Theorem 2</u> Let $[g_{jk}(x)]_{j,k=0}^{n}$ and $[\hat{g}_{jk}(\hat{x})]_{j,k=0}^{n}$ be two Gordon metrics in domains Ω and $\hat{\Omega}$ respectively. Consider two initial-boundary value problems of the form (10) in $\Omega \times \mathbb{R}$ and $\hat{\Omega} \times \mathbb{R}$ respectively, where $\Omega = \Omega_0 \setminus \bigcup_{j=1} \Omega_j$, $\hat{\Omega} = \Omega_0 \setminus \bigcup_{j=1} \hat{\Omega}_j$.

Assume that the refraction indexes n and $\hat{n}$ are constant and that the flow $w(x)$ satisfies (12).

Assume also that (11) hold for both metrics

Then $\Lambda = \hat{\Lambda}$ on $\partial\Omega_0 \times \mathbb{R}$ implies that

$$\hat{n} = n, \quad \hat{\Omega} = \Omega \quad \text{and} \quad \hat{w}(x) = w(x)$$

3. The propagation of light in slowly moving medium

In this case one drops the terms of order $\frac{|w|^2}{c^2}$.
Then the metric tensor has the form

(13) $g^{jk}(x) = \delta^{jk}$ for $1 \le j, k \le n$, $n = 3$,

$$g^{00}(x) = n^2(x), \quad g^{0j}(x) = g^{j0}(x) = (n^2(x) - 1)\frac{w_j(x)}{c}, \ 1 \le j \le n$$

Denote $v_j(x) = g^{0j} = g^{j0}$. We say that the flow $v = (v_1, \ldots, v_n)$ is a gradient flow if $v(x) = \frac{\partial b(x)}{\partial x}$ where $b(x) \in C^\infty(\overline{\Omega})$, $b(x) = 0$ on $\partial\Omega_0$

<u>Theorem 3</u> Consider two initial-boundary value problems in domains $\Omega \times \mathbb{R}$ and $\hat{\Omega} \times \mathbb{R}$ for operators of the form (1) with metrics $[g^{jk}(x)]$, $[\hat{g}^{jk}(\hat{x})]$ of the form (13). Assume that the DN operators Λ and $\hat{\Lambda}$ are equal on $\partial\Omega_0 \times \mathbb{R}$. Assume that there exists an open connected and dense $O \subset \Omega$ such that $v(x)$ does not vanish on O. Then

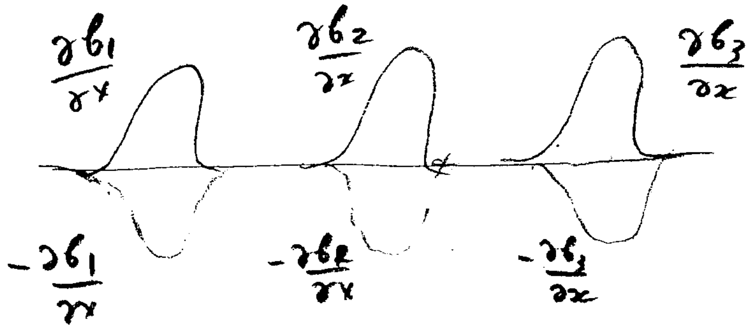
$$\hat{\Omega} = \Omega, \quad \hat{n}(x) = n(x) \text{ and } \hat{v}(x) = v(x)$$

if $v(x)$ is not a gradient flow. In the case of the gradient flow there are two solutions of the inverse problem:

$$\hat{v}(x) = v(x) \quad \text{and} \quad \hat{v}(x) = -v(x)$$

Remark 2. Suppose that the open set O where $b(x) \neq 0$ consists of several open components $O_1, \dots, O_r$. Suppose there exists $b_j(x) \in C^\infty(\bar{\Omega})$, $b_j(x) = 0$ on $\partial\Omega$, $\frac{\partial b_j}{\partial x} = b(x)$ on O_j, $b_j = 0$ in $\bar{\Omega} \setminus O_j$, $j = 1, 2, \dots, r$.

Then we have 2^r solutions of the inverse problem where each of these solution is equal to either $\frac{\partial b_j}{\partial x}$ or to $-\frac{\partial b_j}{\partial x}$ on O_j



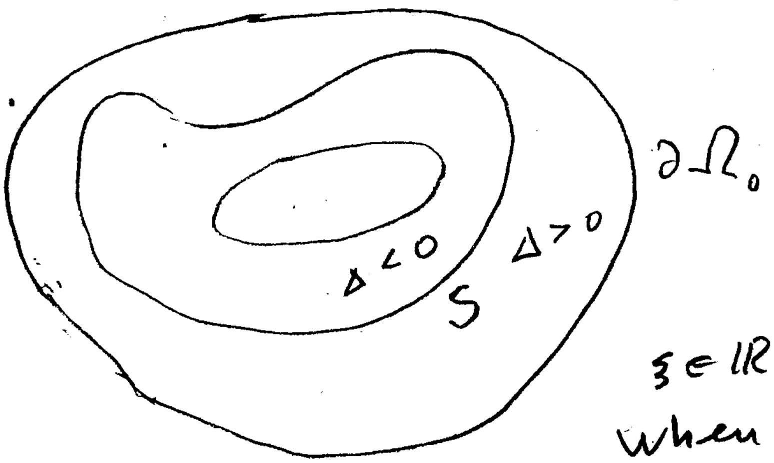
4. Optical black holes

Black holes for the equation (1) are called artificial black holes or optical (acoustic) black holes to emphasize that these are not the black holes for the Einstein equations of the general relativity. Let

$$(1)\qquad \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_j}\left(\sqrt{(-1)^n g(x)}\, g^{jk}(x) \frac{\partial u(x_0, x)}{\partial x_k}\right) = 0,$$

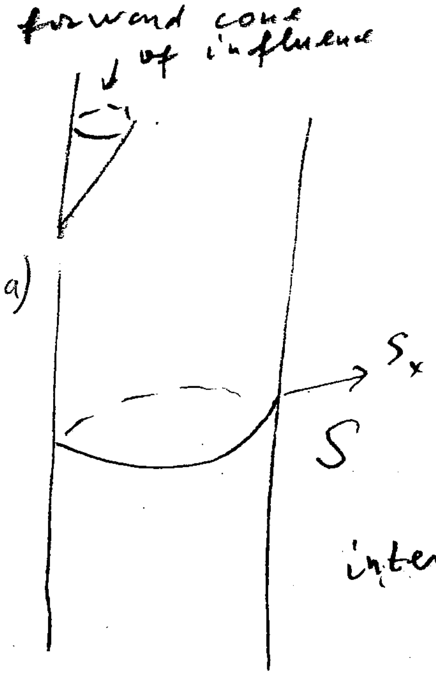
where $g(x) = \det\left(\left[g^{jk}(x)\right]_{j,k=0}^{n}\right)^{-1}$, coefficients in (1) are independent of x_0

Denote $\Delta(x) = \det\left[g^{jk}(x)\right]_{j,k=1}^{n}$

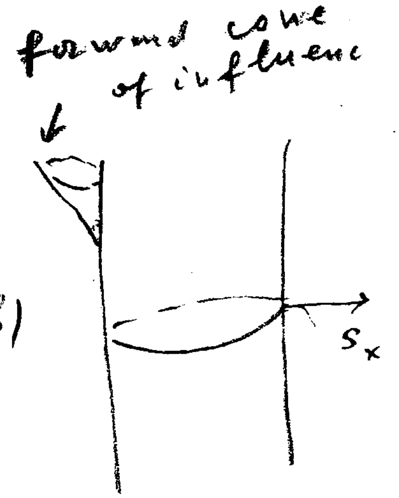


S is the surface where $\Delta(x) = 0$. Note that in the region $\Delta > 0$ any direction $(0, \xi)$, $\xi \in \mathbb{R}^n \setminus \{0\}$, is not characteristic. When $\Delta = 0$ there is a characteristic direction.

In general relativity the surface S where $\Delta = 0$ is called the ergosphere.



black hole



white hole

Consider first the case when the ergosphere S is a characteristic surface i.e.

$$(14) \qquad \sum_{j,k=1}^{n} g^{jk}(x) S_{x_j} S_{x_k} = 0 \quad \text{when } S(x)=0$$

Then the interior of $S\times\mathbb{R}$ is a black or a white hole. In the case a) the interior of $S\times\mathbb{R}$ is a black hole since no signals from inside $S\times\mathbb{R}$ can reach the outside region. In the case b) the interior of $S\times\mathbb{R}$ is a white hole since no signals from the outside of $S\times\mathbb{R}$ can penetrate inside.

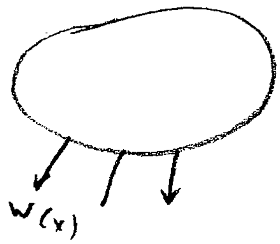
Always when S is a closed characteristic surface in $\mathbb{R}^n$ the interior of $S\times\mathbb{R}$ is either a black or a white hole. Let S_x be the exterior normal to S. In the case when S is an ergosphere (i.e. $\Delta(x)=0$) and (14) holds we have that interior of $S\times\mathbb{R}$ is a black hole if

$$\sum_{j=1}^{n} g^{0j}(x) S_{x_j} < 0 \quad \text{on } S$$

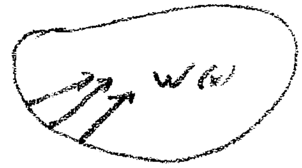
and the interior of $S\times\mathbb{R}$ is a white hole if

$$\sum_{j=1}^{n} g^{0j}(x) S_{x_j} > 0 \quad \text{on } S$$

a) white hole



b) black hole



In the case of the Gordon equation the equation of the ergosphere S is

$$|w(x)|^2 = \frac{c^2}{n^2(x)}$$

S is characteristic surface if the velocity field $w(x)$ is orthogonal to S.

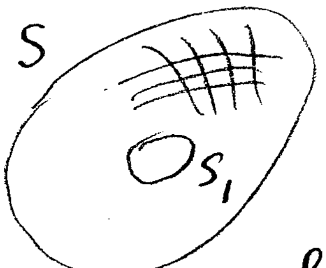
We have a black hole if $w(x)$ is pointed inside S and a white hole if $w(x)$ is pointed outside S.

Note that the black and white holes described above are not stable.
If we perturb metric $[g^{jk}(x)]$ the ergosphere S may cease to be a characteristic surface and the interior of $S \times \mathbb{R}$ will be not a black or white hole.

We describe next the conditions when stable black or white holes exist.

Consider the case $n=2$, i.e. the case of two space variables $x=(x_1,x_2)$

Let S be the ergosphere i.e. $\Delta(x) = g^{11}g^{22} - g^{12}(x) = 0$ on S. Let S_1 be inside S. Denote by $\Omega_e < \Omega$ (ergoregion) the domain between S and S_1. We assume that that $\Delta(x) < 0$ on $\overline{\Omega}_e \setminus S$.



Since $n=2$ there are (locally) two families of characteristic curves satisfying $\sum_{j,k=1}^{2} g^{jk}(x) S^{\pm}_{x_j} S^{\pm}_{x_k} = 0$

We can define globally in Ω_e two family $f^{\pm}(x)$ of nonvanishing vector fields such that

$f^{\pm}(x)$ are tangent to $S^{\pm}(x) = \text{const}$,

$f^{\pm}(x) \neq (0,0)$ on $\overline{\Omega}_e$, $f^{+}(x) \neq f^{-}(x)$ on $\overline{\Omega}_e \setminus S$

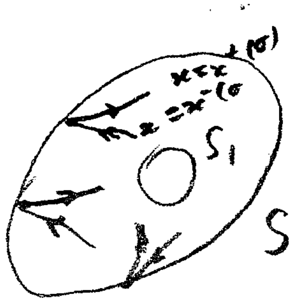
Consider two systems in Ω_e

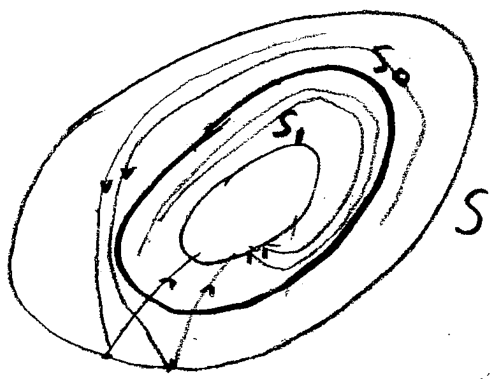
(15) $\dfrac{dx^{+}(\sigma)}{d\sigma} = f_{+}(x^{+}(\sigma))$, $\sigma \geq 0$, $x^{+}(0) = y \in S$,

(15') $\dfrac{dx^{-}(\sigma)}{d\sigma} = f_{-}(x^{-}(\sigma))$, $\sigma \geq 0$, $x^{-}(0) = y \in S$

Note that the trajectories $x = x^{+}(\sigma)$ and $x = x^{-}(\sigma)$ are projections on (x_1,x_2)-plane (up to a reparametrization) of some null-bichoracteristics of the equation (1). Therefore one can use x_0 as a parameter for (15), (15')

It appears that one of the trajectories (say $x = x^{+}(\sigma(x_0))$ starts at $y \in S$ when x_0 is increasing and the another trajectory ($x = x^{-}(\sigma(x_0))$) ends on S when x_0 is increasing





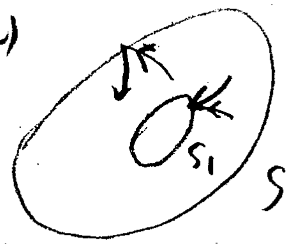
Theorem 4 Assume that the ergosphere S is not characteristic for all $x \in S$ i.e.

$$\sum_{j,k=1}^{2} g^{jk}(x) S_{x_j} S_{x_k} \neq 0 \quad \text{for all } x \in S$$

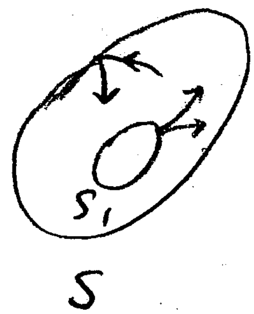
Assume that the projections of all null-bicharacteristics passing through any point $y \in S_1$ are directed either inside S_1 or outside S_1 when x_0 increases. Then there exists $S_0(x)=0$ between S and S_1 such that the interior of $S_0 \times \mathbb{R}$ is either white or black hole.

The proof of Theorem 4 is based on the Poincare-Bendixson theorem

a)



b)



In the case a) $x = x^-(\sigma)$, $x^-(0) = y \in S$ can not reach S_1. In the case b) $x = x^+(\sigma)$, $x^+(0) = y \in S$ can not reach S_1. In both cases by the Poincare-Bendixson theorem there exists a limit cycle $S_0(x) = 0$ and S_0 is a characteristic curve. Therefore the interior $S_0 \times \mathbb{R}$ is either black or white hole.

Applying Theorem 4 to the Gordon equation we get

<u>Theorem 5</u> Let S be the curve $|w|^2 = \frac{c^2}{h^2(x)}$. Suppose $w(x)$ is not colinear with the normal to S for any $x \in S$. Suppose we have

$$(16) \qquad (n^2(x)-1)^{\frac{1}{2}} \left(v(x) \cdot N(x)\right) > 1 \quad \text{on } S_1,$$

or

$$(16') \qquad (n^2(x)-1)^{\frac{1}{2}} \left(v(x) \cdot N(x)\right) < -1 \quad \text{on } S_1,$$

where $v(x) = \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}} \frac{w(x)}{c}$, $N(x)$ is the outword unit normal to S_1

Then there exists a limit cycle $S_0(x) = 0$

<u>Remark 2</u> Note that the black or white holes obtained by Theorems 4 and 5 are stable.

The next theorem specify whether the interior of $S_0 \times \mathbb{R}$ a black or white hole.

<u>Definition</u>: We say that the flow $w(x)$ is incoming if for any closed Jordan curve Γ in Ω_e containing S_1 there exists at least one point $y \in \Gamma$ such that $w(y) \cdot v(y) > 0$ where $v(y)$ is a normal to Γ pointed inside Γ. Analogously the flow $w(x)$ is outgoing if for any Γ there is $y \in \Gamma$ such that $w(y) \cdot v(y) < 0$

<u>Theorem 6</u> Suppose $w(x)$ is incoming and (16) holds. Then the interior of $S_0 \times \mathbb{R}$ is a black hole. If $w(x)$ is outgoing and (16') holds then the interior of $S_0 \times \mathbb{R}$ is a white hole

Example (M. Visser) Acoustic black hole

Consider a fluid flow with the velocity field

$$v = (v^1, v^2) = \frac{A}{r}\,\hat{r} + \frac{B}{r}\,\hat{\theta},$$

where $r = |x|$, $\hat{r} = \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}\right)$, $\hat{\theta} = \left(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\right)$,

A and B are constants. The inverse of metric tensor has the following form in this case: c is the sound speed, ρ is the density.

$$g^{00} = \frac{1}{\rho c}, \quad g^{0j} = g^{j0} = \frac{1}{\rho c} v^j, \quad 1 \le j \le 2,$$

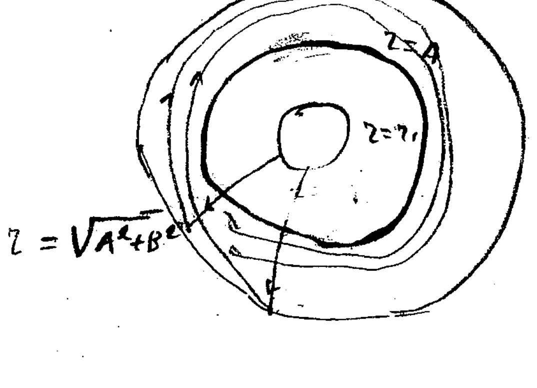
$$g^{jk} = \frac{1}{\rho c}\left(-c^2\delta_{ij} + v^j v^k\right), \quad 1 \le j, k \le 2.$$

Assume $\rho = c = 1$. The consider the case $A > 0$, $B > 0$. Ergosphere is $r = \sqrt{A^2+B^2}$. Consider the domain $\Omega_e = \{r_1 \le r \le \sqrt{A^2+B^2}\}$ where $r_1 < A$. In polar coordinate (r, θ) the differential equations (15), (15') have the form

$$\frac{dr}{ds} = A^2 - r^2, \quad \frac{d\theta}{ds} = \frac{AB}{r} + \sqrt{A^2+B^2-r^2}$$

and

$$\frac{dr}{ds} = -1, \quad \frac{d\theta}{ds} = \frac{1 - \frac{B^2}{r^2}}{\frac{AB}{r} + \sqrt{A^2+B^2-r^2}}$$



$r = A$ is the limit cycle

$\{r = A\} \times \mathbb{R}$ is the boundary of a white hole

A more general example

Consider a fluid flow with the velocity

$$v = A(r)\hat{r} + B(r)\hat{\theta},$$

where $r_1 \le r \le r_0$, $A(r), B(r)$ are smooth, $B(r) > 0$,
$A^2(r_0) + B^2(r_0) = 1$, $A^2(r) + B^2(r) > 1$ on $[r_1, r_0)$,
$A(r)+1$ has simple zeros $\alpha_1, \dots, \alpha_{m_1}$ on (r_1, r_0)
$A(r)-1$ has simple zeros $\beta_1, \dots, \beta_{m_2}$ on (r_1, r_0)
$\alpha_j \ne \beta_k$, $\forall j, \forall k$, $|A(r_1)| > 1$
Here $r = r_0$ is the ergosphere. The differential equations (15), (15') have the following form in polar coordinates (r, θ):

$$\frac{dr}{ds} = A(r) - 1, \quad \frac{d\theta}{ds} = \frac{A(r)B(r) + \sqrt{A^2(r) + B^2(r) - 1}}{A(r)+1}$$

and

$$\frac{dr}{ds} = A(r) + 1, \quad \frac{d\theta}{ds} = \frac{A(r)B(r) - \sqrt{A^2(r) + B^2(r) - 1}}{A(r)-1}$$

Here $r = \alpha_j$, $1 \le j \le m_1$ and $r = \beta_k$, $1 \le k \le m_2$ are limit cycles and there are $m_1 + m_2$ black and white holes.

Axially symmetric metrics

Consider the equation (1) in $\Omega \times \mathbb{R}$ where Ω is a three-dimensional domain. Let (r, θ, φ) be the spherical coordinates in $\mathbb{R}^3$.

Suppose g^{jk} are independent of φ. Consider a characteristic surface S independent of φ and x_0, i.e. S depends on r and θ only. Then S satisfies an equation

$$(17) \quad a^{11}(r,\theta)\Big(\frac{\partial S}{\partial r}\Big)^2 + 2a^{12}(r,\theta)\frac{\partial S}{\partial r}\frac{\partial S}{\partial \theta} + a^{22}(r,\theta)\Big(\frac{\partial S}{\partial \theta}\Big)^2 = 0$$

We assume that $a^{jk}(r,\theta)$ are independent of φ, $1 \le j,k \le 2$.

Consider (17) in two-dimensional domain ω where $\delta_1 \le r \le \delta_2$, $0 < \delta_3 < \theta < \pi - \delta_4$ when $(r,\theta) \in \omega$. Here $\delta_j > 0$, $1 \le j \le 4$.

Assuming that ω and $a^{jk}(r,\theta)$, $1 \le j,k \le 2$, $(r,\theta) \in \omega$ satisfy the condition of Theorem 4 we can prove the existence of black or white holes whose boundary is $S_0 \times S^1 \times \mathbb{R}$ where $\varphi \in S^1$, $x_0 \in \mathbb{R}$ and S_0 is a Jordan curve in ω.

Work in progress (joint with Jim Ralston)

Black holes in the case of time-dependent metrics

Sketch of the proof of Theorem 1

Consider equations $L^{(i)} u_i = 0$ in $\Omega_i \times \mathbb{R}$, $\partial\Omega_1 \cap \partial\Omega_2 \supset \Gamma$, $i=1,2$. We shall show that if $\Lambda^{(1)} = \Lambda^{(2)}$ on $\Gamma \times \mathbb{R}$ then there exists a map Φ of the form

$$\hat{x}_0 = x_0 + a(x),$$

$$\hat{x} = \varphi(x), \quad a(x) = 0 \text{ on } \Gamma, \text{ such that}$$

where $\varphi(x) = x$ on Γ,

$$\Phi \circ L^{(2)} = L^{(1)} \text{ in } \Omega$$

Step I Introduction of "Goursat" coordinates

Let $\Gamma_1 \subset \Gamma$. Introduce coordinates such that Ω near Γ_1 the equation of Γ_1 is $x_n = 0$ and $x_n > 0$ in Ω.
Let $\psi_i^{\pm}(x_0, x)$ be the solutions of the eiconal equations

$$\sum_{j,k=0}^{n} g_i^{jk}(x)\, \psi^{\pm}_{i x_j}(x_0,x)\, \psi^{\pm}_{i x_k}(x_0,x) = 0 \quad \text{for } x_n > 0,\ i=1,2$$

$$\psi_i^{+} = x_0 \quad \text{when } x_n = 0$$

$$\psi_i^{-} = T_1 - x_0 \quad \text{when } x_n = 0, \quad T_1 \text{ is small.}$$

Denote by $\varphi_{ip}(x)$ the solution of

$$\sum_{j,k=0}^{n} g_i^{jk}(x)\, \varphi_{p x_j}\, \psi^{-}_{x_k} = 0, \quad x_n > 0$$

$$1 \le p \le n-1, \quad i = 1,2$$

$$\varphi_p = x_p \quad \text{when } x_n = 0,$$

Make the changes of variables $(i=1,2)$ near $\Gamma_1\times[0,T_1]$:

$$s=\psi_i^+(x_0,x)=x_0+\varphi_i^+(x)$$
$$\tau=\psi_i^-(x_0,x)=T-x_0+\varphi_i^-(x)$$
$$y_j=\varphi_{ij}(x)\ ,\quad 1\le j\le n-1\ ,\quad i=1,2$$

We shall call coordinates (s,τ,y') the Goursat coordinates, $y'=(y_1,\dots,y_{n-1})$

Denote
$$y_n=\frac{T-s-\tau}{2}=-\frac{\varphi_i^+(x)+\varphi_i^-(x)}{2}$$
$$y_0=\frac{s-\tau+T}{2}=x_0+\frac{\varphi_i^+(x)+\varphi_i^-(x)}{2}$$
$$y_j=\varphi_{ij}(x)\ ,\quad 1\le j\le n-1\ ,\quad i=1,2$$

In (y_0,y_n,y') coordinates the equations $L^{(i)}u_i=0$ have the form

$$L_1^{(i)}u_i'\overset{\text{def}}{=}u'_{i\,y_0^2}-u'_{i\,y_n^2}+\sum_{j,k=1}^{n}\frac{\partial}{\partial y_j}\left(g_{i0}^{jk}\frac{\partial u_i'}{\partial y_k}\right)+$$
$$+\sum_{j=1}^{n-1}\left(\frac{\partial}{\partial y_0}-\frac{\partial}{\partial y_n}\right)\left(g_{i0}^{0j}\frac{\partial u_i'}{\partial y_j}\right)+$$
$$+\sum_{j=1}^{n-1}\frac{\partial}{\partial y_j}\left(g_{i0}^{0j}\left(\frac{\partial}{\partial y_0}-\frac{\partial}{\partial y_n}\right)u_i'\right)+Vu_i'=0\ ,\quad i=1,2$$

Here $u_i'=b\,u_i$, $b\neq 0$ and V are some functions

The DN operators have the following form in (y_0, y', y_n) coordinates

$$\Lambda_1^{(i)} f\Big|_{y_n=0} = \left(\frac{\partial u_i'}{\partial y_n} + \sum_{j=1}^{n-1} g_{i0}^{j0} \frac{\partial u_i'}{\partial y_j}\right)\Big|_{y_n=0}, \quad i=1,2$$

where $\Lambda_1^{(i)}$ corresponds to $L_1^{(i)}$, $i=1,2$.

Note that $\Lambda^{(1)} = \Lambda^{(2)}$ on $\Gamma \times \mathbb{R}$ implies that $\Lambda_1^{(1)} = \Lambda_1^{(2)}$ on $\Gamma_1 \times (0, T_1)$

Step II Green's formula

Let f and g be arbitrary smooth functions with supports in $\Gamma_1 \times (0, T_1]$. Denote by u_i^f, v_i^g, $i=1,2$ the solutions of the following initial-boundary value problems in $y_n > 0$:

$$L_1^{(i)} u_i^f = 0 \quad \text{for } y_n > 0,$$
$$u_i^f = 0 \text{ for } y_0 < 0, \quad u_i^f\Big|_{y_n=0} = f, \quad i=1,2,$$

$$L_1^{(i)} v_i^g = 0 \quad \text{for } y_n > 0,$$
$$v_i^g = 0 \text{ for } y_0 < 0, \quad v_i^g\Big|_{y_n=0} = g$$

From $0 = (L_1^{(i)} u_i^f, v_i^g) - (u_i^f, L_1^{(i)} v_i^g)$ we get the following Green's formula

$$\int_{\tau=0} \left(\frac{\partial u_i^f}{\partial s} \overline{v_i^g} - u_i^f \overline{\frac{\partial v_i^g}{\partial s}}\right) ds\, dy' = \int_{\Gamma_1 \times (0,T_1)} \left(\Lambda_1^{(i)} f \overline{g} - f \overline{\Lambda_1^{(i)} g}\right) dy_0\, dy'$$

Since $\Lambda_1^{(1)} = \Lambda_1^{(2)}$ we get that

$$\int_{\tau=0} \left(u_{1s}^f \overline{v_1^g} - u_1^f \overline{v_{1s}}\right) ds\, dy' = \int_{\tau=0} \left(u_{2s}^f \overline{v_2^g} - u_2^f \overline{v_{2s}^g}\right) ds\, dy'$$

Since $\int_{\tau=0} u_i^f \overline{v_{is}^g}\, ds\,dy' = -\int_{\tau=0} u_{is}^f \overline{v_i^g}\, ds + \int_{\substack{\tau=0\\ y_n=0}} u_i^f \overline{v_i^g}\, dy'$

and $u_i^f\big|_{y_n=0} = f$, $v_i^g\big|_{y_n=0} = g$

we get that

(18) $\int_{\tau=0} u_{1s}^f \overline{v_1^g}\, ds\,dy' = \int_{\tau=0} u_{2s}^f \overline{v_2^g}\, ds\,dy'$

Step III Important lemma

It follows from (18) that

$\int_{\tau=0} u_s^f \overline{v^g}\, ds\,dy'$

is an invariant of the boundary measurements i.e. it is uniquely determined by the DN operator. The following lemma holds:

Lemma For any $s_0 \in (0, T_1)$

(19) $\int_{\substack{\tau=0\\ s\geq s_0}} u_{1s}^f \overline{v_1^g}\, ds\,dy' = \int_{\substack{\tau=0\\ s\geq s_0}} u_{2s}^f \overline{v_2^g}\, ds\,dy'$

Therefore $\int_{\substack{\tau=0\\ s\geq s_0}} u_s^f \overline{v^g}\, ds\,dy'$ is also an invariant of the boundary measurements. The importance of this invariant is that v^g and u_s^f extended by zero for $s < s_0$ have a jump when $s = s_0$. Then an application of geometric optics solutions allows to pick up the principal term when $s = s_0$, $k \to \infty$.

I learned the idea of the use of boundary measurement invariant with a jump from the the BC (Boundary Control) method (Belishev, Belishev - Kurylev, Kurylev - Lassas)

Step IV The use of the geometric optics solutions

We look for u_i^f in the form

(20) $$u_i^f = e^{ik(s-s_0)} \sum_{p=0}^{N} \frac{1}{(ik)^p} a_p^{(i)}(s,\tau,y') + u_i^{(N+1)},$$

where k is a large parameter,

$$\frac{\partial a_0^{(i)}}{\partial \tau} - \sum_{j=1}^{n-1} g_{i0}^{oj} \frac{\partial a_0^{(i)}}{\partial y_j} = 0$$

$$f = a_0^{(i)}\Big|_{y_n=0} = \chi_1(s)\chi_2(y'),$$

$\chi_1(s) \in C_0^\infty(\mathbb{R}^1)$, $\chi_1(s)=1$ for $|s-s_0|<\delta$, $\chi_1(s)=0$ for $|s-s_0|>2\delta$, $\chi_2(y') \in C_0^\infty(\Gamma_1')$, $\Gamma_1' \subset \Gamma_1$. Note that $a_0^{(i)}(s,\tau,y')$ has the form

$$a_0^{(i)} = \chi_1(s)\chi_2\left(\alpha^{(i)}\left(\frac{T_1-s-\tau}{2}, y'\right)\right)$$

where $\alpha^{(i)} = (\alpha_1^{(i)},\dots,\alpha_{n-1}^{(i)})$,

$$\frac{\partial \alpha_j^{(i)}\left(\frac{T_1-s-\tau}{2},y'\right)}{\partial \tau} - \sum_{j=1}^{n-1} g_{i0}^{jo} \frac{\partial \alpha_j^{(i)}}{\partial y_j} = 0, \quad \alpha_j^{(i)}\Big|_{y_n=0} = y_j,$$

$1 \le j \le n-1$

Consider the change of variables

(21) $$\hat{s} = s,\ \hat{\tau} = \tau,\ \hat{y} = \alpha^{(i)}\left(\frac{T_1-s-\tau}{2}, y'\right),\quad i=1,2.$$

Let

(22) $$s = \hat{s},\ \tau = \hat{\tau},\ y' = \beta^{(i)}\left(\frac{T_1-s-\tau}{2}, \hat{y}'\right),\quad i=1,2$$

be the inverse change of variables. Substituting (20) into (19) and taking the limit when $k \to \infty$ we get after some computations that

(23) $$v_1^{j}\left(s,0,\beta^{(1)}\left(\frac{T_1-s}{2},\hat{y}\right)\right) = v_2^{j}\left(s,0,\beta^{(2)}\left(\frac{T_1-s}{2},\hat{y}'\right)\right)$$

Step V The conclusion of the proof of Theorem 1

Changing in (19) $\tau = 0$ to $\tau = \tau'$, $0 \le \tau' \le T_1$ we obtain analogously to (23)

$$v_1^g(-s, \tau, \beta^{(1)}(\tfrac{T-s-\tau}{2}, \hat{y}')) = v_2^g(s, \tau, \beta^{(2)}(\tfrac{T_1-s-\tau}{2}, \hat{y}'))$$

Denote

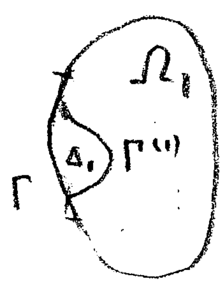
$$w_i^g(s, \tau, \hat{y}') = v_i^g(s, \tau, \beta^{(i)}(\tfrac{T-s-\tau}{2}, \hat{y}')), \quad i = 1, 2$$

Then $\quad w_1^g(s, \tau, \hat{y}') = w_2^g(s, \tau, \hat{y}')$

Let $\tilde{L}_1^{(i)}$ be the operators $L_1^{(i)}$ after the change of coordinates (21). Then

$\tilde{L}_1^{(1)} w_1^g = 0$, $\tilde{L}_1^{(2)} w_1^g = 0$

i.e. $\tilde{L}_1^{(1)}$ and $\tilde{L}_1^{(2)}$ have the same solutions. One can proof that $\{w_1^g\}$, $\forall g \in \mathring{H}_1(\Gamma_1 \times (0, T_1))$, are dense in $H^1(R_{\tau'})$ where $R_{\tau'}$ is a rectangle in the plane $\tau = \tau'$. This implies that the coefficients of $\tilde{L}_1^{(1)}$ and $\tilde{L}_1^{(2)}$ are equal.



Combining the changes of variables in Steps I–V we get that there exists a subdomains $\Delta_1 \subset \Omega_1$ and $\Delta_2 \subset \Omega_2$ such and a diffeomorphism φ of $\Delta_2 \times \mathbb{R}$ onto $\Delta_1 \times \mathbb{R}$ such that $\varphi \circ L_2 = L_1$ on $\bar{\Delta}_1 \times \mathbb{R}$. We assume that $\partial \Delta_1 \cap \partial \Omega_1 = \partial \Omega_2 \cap \partial \Omega_2 \subset \Gamma$, $\varphi = I$ on $(\partial \Omega_1 \cap \Gamma) \times \mathbb{R}$.

Extend φ from $\bar{\Delta}_2 \times \mathbb{R}$ to $\bar{\Omega}_2 \times \mathbb{R}$, $\varphi = I$ on $\bar{\Gamma} \times \mathbb{R}$. Denote this extension by φ_3. Let $\bar{\Omega}_3 \times \mathbb{R} = \varphi_3(\bar{\Omega}_2 \times \mathbb{R})$ and $L^{(3)} = \varphi_3 \circ L_2$. We have that $\bar{\Delta}_1 \subset \bar{\Omega}_3$, $L^{(3)} = L^{(1)}$ on Δ_1. One can prove that $\Lambda^{(1)} = \Lambda^{(2)}$ on $\Gamma \times \mathbb{R}$ implies that $\tilde{\Lambda}^{(1)} = \tilde{\Lambda}^{(3)}$ on $\Gamma^{(1)} \times \mathbb{R}$ where $\tilde{\Lambda}^{(1)}, \tilde{\Lambda}^{(3)}$ are DN operators for $L^{(1)}$ and $L^{(3)}$ on smaller domains $(\Omega_1 - \bar{\Delta}_1) \times \mathbb{R}$ and $(\Omega_3 - \bar{\Delta}_1) \times \mathbb{R}$ respectively

Then we repeat the same arguments for $L^{(1)}$ and $L^{(3)}$ starting with smaller domains $\Omega_1 \setminus \bar{\Delta}_1$ and $\Omega_3 \setminus \bar{\Delta}_1$, etc.

One can show that in a finite number of steps we exhaust the domain Ω_1