Monge Ampère geodesics and semi-classical approximations

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Based on joint work (partly in progress) with Yanir Rubinstein; and Jian Song

Topic and theme

Our goal is to use semi-classical methods to approximate solutions of the HCMA equation = homogeneous complex Monge Ampère equation on $A \times M$ where $A \subset \mathbb{C}$ is an annulus and M is a Kähler manifold.

For the Cauchy problem, there is no other general method.

I.e. we use geometric quantization to solve a fully nonlinear equation.

Semi-classical methods must be enlarged to include large deviations techniques.

Setting

Let

- (M, ω_0) = Kähler manifold; most problems already hard for $\mathbb{CP}^m$, $\mathbb{C}^m/\Lambda$ (Abelian variety);
- $\omega_0 \in H^{(1,1)}(M, 2\pi i\mathbb{Z})$.
- $L \to M$ with $\omega_0 \in c_1(L)$.

The space of Kähler metrics in a fixed Kähler class

Consider the space of all Kähler metrics in the same cohomology class:

$$\mathcal{H} \;=\; \{\omega \in H^{(1,1)}(M) : \omega \in c_1(L), \omega >> 0\ \}\ .$$

By the $\partial\bar{\partial}$ Lemma,

$$\mathcal{H} \;=\; \{\varphi \in C^\infty(M) : \omega_\varphi \;=\; \omega_0 + i\partial\bar{\partial}\varphi > 0\ \}\ .$$

$\mathcal{H}$ is an infinite dimensional symmetric space

Equip $\mathcal{H}$ with the L^2 Riemannian metric

(1)

$$||\psi||_{\varphi}^2 = \int_M |\psi|^2 \frac{\omega_{\varphi}^m}{m!}, \quad \text{where } \varphi \in \mathcal{H} \text{ and } \psi \in T_{\varphi}\mathcal{H}.$$

Then $(\mathcal{H}, ||\psi||_{\varphi}^2)$ is a formally an infinite dimensional symmetric space of non-positive curvature (Mabuchi, Semmes, Donaldson).

Formally, $\mathcal{H} = \mathcal{G}_{\mathbb{C}}/\mathcal{G}$ where $\mathcal{G} = Diff_{\omega_0}(M)$ is the group of ω_0 symplectic diffeomorphisms and $\mathcal{G}_{\mathbb{C}}$ is its (formal) complexification.

Geodesics and homogeneous Monge-Ampère

The geodesics of $\mathcal{H}$ are paths φ_t in $\mathcal{H}$ which solve

$$\ddot{\varphi} - |\partial\dot{\varphi}|_{\omega_{\varphi}}^2 = 0. \quad (2)$$

The geodesic equation is equivalent to the HCMA = complex homogeneous Monge-Ampère equation

$$(\omega_0 + \frac{i}{2}\partial\bar{\partial}\Phi)^{m+1} = 0, \quad \text{on } R \times M,$$

where $\Phi(z, w) = \varphi_{\log|w|}(z)$ and where R could be the annulus $A = \{w \in \mathbb{C} : 1 \leq |w| \leq e\}$ or the unit disc D and where Φ is invariant under rotations (Semmes; Donaldson; good results of Phong-Sturm).

Why study geodesics? (see Phong's Lectures)

The geometry of $\mathcal{H}$ is relevant to the study of the relations between

1. Stability of the polarized Kähler manifold (M, ω_0, L).

2. Existence of canonical metrics in $[\omega_0]$, i.e. metrics of constant scalar curvature.

The first is an algebro-geometric notion, the second is differential geometric. Donaldson and others are developing infinite dimensional analogues of GIT to relate (1) and (2). Geodesics are the infinite dimensional 1 PS (one-parameter subgroups), i.e. they are formally the 1 PS of $\mathcal{G}_{\mathbb{C}}$ (which does not exist).

Initial value or endpoint problem

There are two natural problems for geodesics:

1. the endpoint problem of finding a geodesic segment between two given metrics; and

2. the initial value or Cauchy problem, which corresponds to prescribing ω_{φ_0} and an initial velocity $\dot{\varphi}_0$.

The endpoint problem is a Dirichlet boundary problem for the Monge-Ampère equation on $A \times M$, and falls into a classical framework. But the initial value problem is ill-posed. Yet it is geometrically important – it defines the exponential map of $\mathcal{H}$. Geodesic rays are important in Kähler geometry.

Regularity results

For the Dirichlet (endpoint) problem, if the boundary data $\varphi_0, \varphi_1 \in C^\infty$, then $\varphi_t \in C^{1,\alpha}(R\times M)$ for all $\alpha < 1$ (X. X. Chen; Bo Guan; J.Spruck). It is not understood whether one should expect smoother solutions.

Almost nothing is known about the Cauchy problem.

Semmes-Donaldson's formal solution of the initial value problem

Let $\exp tH_{\dot\varphi_0}$ denote the Hamiltonian flow of $\dot\varphi_0$ with respect to ω_{φ_0}, and let $\exp itH_{\dot\varphi_0}$ "be" its analytic continuation in time to the Hamiltonian flow at *imaginary time* it.

Formally, the solution φ_t of the initial value problem with initial data $(\varphi_0, \dot\varphi_0)$ is given by

(3) $$\exp itH^*_{\dot\varphi_0}\omega_0 - \omega_0 = i\partial\bar\partial\varphi_t.$$

But there is no obvious reason why a C^∞ Hamiltonian flow $\exp itH_{\dot\varphi_0}$ should admit an analytic continuation in t for any length of time unless the data is analytic (Cauchy-K again).

Semi-classical analysis enters

There are no prior results on solutions of the Cauchy initial value problem for the HCMA equation.

Y. Rubinstein and I propose an approach to solving it by means of 'semi-classical approximations'. But they are non-standard.

It reconciles Donaldson-Semmes' formal Hamiltonian definition with Phong-Sturm's definition of a geodesic ray associated to a 'test configuration'.

Kähler and Toeplitz quantization

- $H^0(M, L^k)$ = holomorphic sections of the kth tensor power of L.

- h = Hermitian metric on L, $h = e^{-\varphi}$ locally.

- $Hilb_k(h)$: Hermitian inner product

$$\langle s, s \rangle = \int_M |s(z)|^2_{h^k} dV_h,$$

  where $dV_h = \omega_h^m / m!$ is the Kähler volume form induced by $\omega_h = dd^c \log h$.

- $\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$ is the orthogonal projection wrt $Hilb_k(h)$ (Bergman).

How to quantize the classical Hamiltonian flow

We first quantize the Hamiltonian $\dot{\varphi}$ with respect to the initial Kähler form ω_0 as the first order Toeplitz operator

$$\Pi_k \dot{\varphi}_0 \Pi_k, \quad (\Pi_{h_0^k} = \Pi_k)$$

on $H^0(M, L^k)$ where $\dot{\varphi}_0$ denotes the multiplication operator by $\dot{\varphi}_0$. Also, $dd^c \log h_0 = \omega_0$.

Definition: The quantization of the Hamilton flow of $\dot{\varphi}$ is the sequence of unitary groups

$$\mathcal{U}_{h_0^k}(t) = \Pi_k e^{itk \Pi_k \dot{\varphi}_0 \Pi_k} \Pi_k \tag{4}$$

In what sense does $\mathcal{U}_{h_0^k}(t)$ quantize $\exp tH_{\dot{\varphi}}$?

The sequence $\{\mathcal{U}_{h^k}(t)\}$ form a semi-classical Fourier integral Toeplitz operator. Or $\sum_{k=1}^{\infty} \mathcal{U}_{h^k}(t)$ is a usual Fourier integral Toeplitz operator (in the sense of Boutet de Monvel and Guillemin).

Let $X = \partial D_h^* \subset L^*$ be the boundary of the unit disc bundle in the Hermitian metric h. Let $\Sigma \subset T^*X$ be the symplectic cone spanned by the connection one form $\alpha = h^{-1}\partial h$.

Proposition **1** *$\mathcal{U}(t)$ is a group of complex Fourier integral operators on $\mathcal{L}^2(X)$ whose underlying canonical relation is the graph of the Hamiltonian flow of $\dot{\varphi}_0$ on the symplectic cone Σ.*

Analytic continuation of $U_k(t)$

$U_k(t)$ admits an analytic continuation in time t and induces the imaginary time semi-group

$$U_k(i\tau) : H^0(M, L^k) \to H^0(M, L^k). \tag{5}$$

However, $U_k(i\tau)$ is in general not a complex Fourier integral operator (or Toeplitz operator), for the same reason that $\exp i\tau H_{\dot\varphi}$ is not well-defined.

But we can define the analytic continuation of $\exp tH_{\dot\varphi_0}$ by taking the classical limit of its quantization. We do this by consider the Schwartz kernel $U_k(i\tau)(z, w)$ of this operator with respect to the volume form $dV_h = \frac{\omega_h^m}{m!}$.

Conjecture

Definition: *Put*

$$\varphi_k(\tau, z) = \frac{1}{k} \log U_k(i\tau, z, z). \tag{6}$$

We then define the quantum analytic continuation potential by

$$\varphi_t = \lim_{\ell \to \infty} \{\sup_{k \geq \ell} \varphi_k(t)\}^*.$$

Conjecture **1** *The one-parameter family φ_t is a weak solution of the Homogeneous Monge-Ampère equation with initial value φ_0 (the potential of ω_0) and initial velocity $\dot\varphi$.*

Main result to date

Theorem **1** *The conjecture is true for toric Kähler manifolds* (M, L, ω).

Sketch of Proof: We explicitly calculate

$$\varphi_t = \lim_{k \to \infty} \varphi_k(t)$$

and verify that it solves the Cauchy problem.

Background on toric Kähler manifolds

A toric variety M^m carries a $(\mathbb{C}^*)^m$ action with an open orbit M_0. Ex: $\mathbb{CP}^1$ only TV of dim 1. Higher D: $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$, Hizebruch surfaces.

A toric Kähler metric ω is invariant under the real $\mathbf{T}^m = (S^1)^m$ action. $\mathbf{T}^m$ is a Hamiltonian action with moment map $\mu_\omega : M \to P$ with image convex Delzant lattice polytope.

$$H^0(M, L^k) = \text{span}\,\{z^\alpha : \alpha \in kP\}.$$

Kähler potential and symplectic potential

On the open orbit M^0, a toric Kähler metric

has a Kähler potential φ invariant under $\mathbf{T}^m$ = a convex function $\varphi(\rho)$ on $\mathbb{R}^m$. Its gradient $\nabla_\rho \varphi = \mu(\rho)$ is the moment map $\mu : M \to P$.

Its Legendre transform

(7) $$u_\varphi(x) = \mathcal{L}\varphi(x) := \sup_\rho(\langle x, \rho\rangle - \varphi(\rho))$$

a convex function on the polytope P of M.

Geodesics of symplectic potentials

The complex Monge-Ampère equation on $\mathcal{H}_{\mathbf{T}^m}$ linearizes to the equation $\ddot{u} = 0$, and the initial value problem is solved by

(8) $$u_t = u_{\varphi_0} + t\dot{u}_0,$$

where $\dot{u}_0 = \frac{d}{dt}|_{t=0}\mathcal{L}\varphi_t$. One has $\dot{\varphi}_0 = -\dot{u}_0 \circ \mu$.

One can verify that $\mathcal{L}(u_0 + t\dot{u}_0)$ is a weak solution of the Monge Ampère equation even when $u_0 + t\dot{u}$ is not convex.

The main point of our result is that the quantization method produces the same solution.

Lifespan and global solution

If we demand that the solution be smooth, then its lifespan equals

$T_{lifespan}(\varphi_0, \dot{\varphi}_0) = \sup\{t \geq 0 : u_0 + t\dot{u} \text{ is convex}\}$.

For $T < T_{lifespan}$, $\varphi_k \to \varphi_t \in C^2([0,T] \times M)$.

There exists a Lipschitz solution for all time, and

$$\varphi_k \to \varphi_t \text{ in } C^0([0,T] \times M).$$

Question: does this result generalize?

Sketch of proof

To prove this, we first simplify the operator $\mathcal{U}_k(t) = \Pi_k e^{itk\Pi_k \dot{\varphi}_0 \Pi_k} \Pi_k$

Definition: *Let*

$$V_k(t) = \Pi_k e^{ik(t\dot{u}_0)(D_\theta |k|^{-1})} \Pi_k,$$

where $D_\theta = (D_{\theta_1}, \ldots, D_{\theta_m})$ *generates the* $(S^1)^m$ *action*

Theorem **2** *The sequence* $\{V_k(t)\}$ *of unitary groups, is a unitary Toeplitz Fourier integral group quantizing the Hamiltonian flow of* $\dot{\varphi}_0$.

The key point is that $\Pi \dot{u}(D_\theta |k|^{-1}) \Pi$ is a Toeplitz operator of order zero and its symbol is given by

$$H(x) = \dot{u}_0(\mu(x)) = \dot{\varphi}(x).$$

Explicit formulae for φ_k

Proposition **2** *With the above notation, we have*

$$V_k(it) = \Pi_k e^{kt\dot{u}_0(\frac{\alpha}{k})} \Pi_k.$$

Thus,

$$\text{(9)} \qquad V_k(it, z, z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} e^{kt\dot{u}_0(\frac{\alpha}{k})} \frac{|z^\alpha|_k^2}{||z^\alpha||_{L^2}}.$$

Hence,

$$\varphi_k(t, z) = \frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} e^{t\dot{u}(\frac{\alpha}{k})} \frac{|z^\alpha|_k^2}{||z^\alpha||_{L^2}}.$$

Proof of theorem

We need to prove that that

$$\frac{1}{k} \log \textstyle\sum_{\alpha \in kP \cap \mathbb{Z}^m} e^{\dot{u}(\frac{\alpha}{k})} \frac{|z^\alpha|^2_{h_0^k}}{||z^\alpha||^2_{L^2}}$$

$$\to \mathcal{L}(u_0 + t\dot{u}) \;\; \forall t \text{ in } \; C^0,$$

$$\to \mathcal{L}(u_0 + t\dot{u}) \;\; \forall t \leq T_{lifespan} \text{ in } \; C^2$$

The proof of the C^2 step is a corollary of earlier work J. Song and S.Z. for the endpoint problem.

Family of probability measures and Varadhan Lemma

Since we do not know in advance the regularity of φ_t, we use a robust method based on the probability measures,

$$(10) \qquad \mu_k^z = \frac{1}{\Pi_k(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|z^\alpha|^2_{h_0^k}}{||z^\alpha||^2_{L^2}} \delta_{\frac{\alpha}{kd}},$$

where

$$\Pi_k(z,z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|z^\alpha|^2_{h_0^k}}{||z^\alpha||^2_{L^2}}$$

is the contracted Szegö kernel on the diagonal (or density of states);

Large deviations principle

Theorem **3** *For any $z \in M$, the probability measures μ_k^z satisfy a uniform Laplace large deviations principle with rate k and with convex rate functions $I^z \geq 0$ on P. defined as follows:*

- *If $z \in M^0$, the open orbit, then $I^z(x) = u_0(x) - \langle x, \log|z|\rangle + \varphi_{P^o}(z)$, where φ_{P^o} is the canonical Kähler potential of the open orbit and u_0 is its Legendre transform, the symplectic potential;*

- *When $z \in \mu_0^{-1}(F)$ for some face F of ∂P, then $I^z(x)$ restricted to $x \in F$ is a restricted version. On complement of $\bar{F}$ it is defined to be $+\infty$.*

(Extra Material - not covered in lecture 10.27.08)

Varadhan's Lemma and C^0 convergence

Varadhan's Lemma *Let $d\mu_k$ be probability measures on X which satisfy the LDP with rate k and rate function I on X. Let F be a continuous function on X which is bounded from above. Then*

$$\lim_{k\to\infty} \frac{1}{k} \log \int_X e^{kF(x)} d\mu_k(x) = \sup_{x\in X} [F(x) - I(x)].$$

This gives C^0 convergence of our ray

$$\varphi_k(t,z) = \frac{1}{k} \log \sum_{\alpha\in kP\cap\mathbb{Z}^m} e^{t\dot{u}(\frac{\alpha}{k})} \frac{||z^\alpha||^2_{h_0^k}}{||z^\alpha||^2_{L^2}}.$$

C^1 convergence analogous to 'absence of phase transitions'.

Background and related results

Our result and conjecture on the solution of the Cauchy problem for the HCMA equation is of the type "Bergman metric approximations." To put it into context, we need to define

- Bergman or Fubini-Study metrics;
- Systematic method of approximating any C^∞ metric by Bergman metrics;
- Phong Sturm initiated approximations to geodesics through Bergman geodesics and used them to define 'test configuration rays.'

Bergman metrics

Let $d_k+1 = \dim H^0(M, L^k)$ and let $\mathcal{B}H^0(M, L^k)$

denote the manifold of all bases $\underline{s} = \{s_0, \dots, s_{d_k}\}$ of $H^0(M, L^k)$. Given a basis, we define the Kodaira embedding

$$\Phi_{\underline{s}} : M \to \mathbb{CP}^{d_k}, \;\; z \to [s_0(z), \dots, s_{d_k}(z)].$$

A Bergman (hermitian) metric of height k is a metric of the form

(11)
$$h_{\underline{s}} := (\Phi_{\underline{s}}^* h_{FS})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}\right)^{1/k}},$$

where h_{FS} is the Fubini-Study Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^{d_k}$. We then define

(12) $$\mathcal{B}_k = \{h_{\underline{s}}, \;\; \underline{s} \in \mathcal{B}H^0(M, L^k)\}.$$

Systematic approximation $\mathcal{H} \to \mathcal{B}_k$

Let h be any C^∞ Hermitian metric on L. Recall that it induces inner products $Hilb_k(h)$ on $H^0(M, L^k)$. We use any orthonormal basis $\mathcal{S}_k$ as a Kodaira map $\Phi_{\mathcal{S}} : M \to \mathbb{CP}^{d_k}$ and pull back the Fubini-Study metric h_{FS}. Thus,

$$Hilb_k : \mathcal{H} \to \mathcal{B}_k, \;\; h \to \; h(k) = (\Phi_{\mathcal{S}_k}^* h_{FS})^{1/k},$$

$\mathcal{S}_k =$ an orthonormal basis of $H^0(M, L^k)$ for h.

The metric $h(k)$ is independent of the choice of orthonormal basis.

Then $h(k) \to h$ in C^∞ and has a complete asymptotic expansion in k^{-1}. (Tian-Yau-Z-(Catlin); Boutet de Monvel-Sjöstrand parametrix).

Bergman Kähler potentials

We defined $\mathcal{H}$ is the space of Kähler potentials of Kähler metrics in the fixed class. The Kähler potential (relative to h_0) corresponding to $h_{\underline{s}}$ is

$$\varphi_{\underline{s}}(z) = \frac{1}{k} \log \sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}. \tag{13}$$

Bergman geodesics

$\mathcal{B}_k \simeq GL(d_k + 1, \mathbb{C})/U(d_k + 1)$ is a symmetric space.

Geodesics in $\mathcal{B}_k$ = 1 PS (one-parameter subgroups) e^{tA} of $GL(d_k, \mathbb{C})$. If $A = Diag(\lambda_j)$ the Bergman geodesic is,

$$\varphi_k(t; z) = \frac{1}{k} \log \left(\sum_{j=0}^{N} e^{2\lambda_j t} |\hat{s}_j^{(0)}(z)|^2_{h_0^k} \right).$$

Ideas for general (M, ω)

For general (M, g) we use *coherent states* $\Phi^w_{h^k}$ as a basis, instead of eigensections of e^{A_k}.

Here, $\Phi^w_{h^k}(z) = \frac{\Pi_{h^k}(z,w)}{\sqrt{\Pi_{h^k}(w,w)}}$ are L^2 normalized Szegö kernels pinned down in the second slot. They are intuitively like Gaussian bumps centered at w. Under the Fourier Toeplitz unitary group $U_{h^k}(t)$ the coherent states propagate without changing their shape, $U_{h^k}(t)\Phi^w_{h^k}(z) \sim \Phi^{\Psi^t(w)}_{h^k}(z)$ where Ψ^t is the Hamilton flow of $\dot{\varphi}$. After analytic continuation, they should propagate and change their shape. We conjecture that $U_{h^k}(it)\Phi^w_{h^k}(z) \sim \Phi^{f_t(w)}_{h^k_t}(z)$, where h_t is the Monge-Ampère geodesic and f_t is the Moser diffeomorphism such that $f_t^*\omega_0 = \omega_t$.

Optimal transport

The Moser path of maps $f_t : M \to M$ such that $f_t^*\omega_0 = \omega_t = \omega_0 + dd^c\varphi_t$ corresponding to a geodesic ω_t is given by:

Theorem **4** *In the case of toric Kähler manifolds, f_t restricted to the real toric variety $M_{\mathbb{R}}$ is given by*

$$f_t = \mu_t^{-1}\mu_0,$$

where $\mu_t = \nabla\varphi_t$.

It may be interpreted as the optimal transport map taking ω_0 to ω_t.