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GENERALIZATION OF SELBERG'S

SPECTRAL GAP THEOREM

PRIME SIEVING IN ORBITS OF LINEAR GROUPS

$\Lambda \subset SL_2(\mathbb{Z})$ non-elementary subgroup

$r(z)=$ number of prime factors of $z \in \mathbb{Z}\backslash\{0\}$

Theorem. (BGS)

There is a constant $C(\Lambda)$ such that

$$\left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda \middle| r(abcd) < C(\Lambda)\right\}$$

is Zariski dense in SL_2.

Theorem. (BGS)

Let $f \in \mathbb{Q}[x_1, x_2, x_3, x_4]$ *taking integer values on* Λ *and not a multiple of*

$$g(x_1, x_2, x_3, x_4) = x_1 x_4 - x_2 x_3 - 1$$

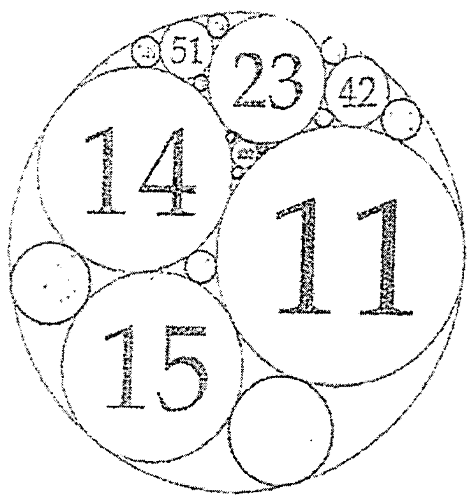
There is $r = r(\Lambda) \in \mathbb{Z}_+$ *s.t.*

$\{x \in \Lambda | f(x)$ *has at most* r *prime factors*$\}$

is Zariski dense in SL_2

EXPLICIT APPLICATIONS

Example. Appolonian packings



Appolonian packing corresponding to:

Quadruple $(-6, 11, 14, 23)$

DESCARTE FORM

$F(x_1, x_2, x_3, x_4) =$

$2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2$

$O_F =$ Orthogonal group

$A = \langle S_1, S_2, S_3, S_4 \rangle$

$=$ Appolonian packing group

$$S_1 = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix}$$

ROLE OF EXPANDER GRAPHS

Let S be a finite subset of $SL_2(\mathbb{Z})$ generating a free group

Theorem. *(BGS)*

There is $q_0 \in \mathbb{Z}$ such that the family of Cayley graphs

$$\mathcal{G}\big(SL_2(\mathbb{Z}/q\mathbb{Z}), \pi_q(S)\big)$$

$$q \text{ square free}, (q, q_0) = 1$$

is a family of expanders

Lubotzky-Weiss Conjecture

Let S be a finite subset of $SL_d(\mathbb{Z})$ generating a Zariski dense subgroup of SL_d. Then there is $q_0 \in \mathbb{Z}$ such that the family of Cayley Graphs

$$\mathcal{G}(SL_d(\mathbb{Z}/q\mathbb{Z}), \pi_q(S))$$

$$q \in \mathbb{Z}_+, (q, q_0) = 1$$

forms a family of expanders

CONNECTEDNESS OF THE GRAPH

ROLE OF STRONG APPROXIMATION PROPERTY

Theorem. Let G be a Zariski dense subgroup of $SL_d(\mathbb{Z})$. There is $q_0 \in \mathbb{Z}$ such that $\pi_q(G) = SL_d(\mathbb{Z}/q\mathbb{Z})$ if $(q, q_0) = 1$

π_q: reduction mod q

Matthews, Vaserstein, Weisfeiler, Pink

KNOWN RESULTS

- $SL_2(p)$ **(B-G)** based on HELFGOTT

- $SL_2(q)$ (q square free) **(B-G-S)**

- $SL_2(p^n)$ **(B-G)**

- $SL_d(p^n)$ (p fixed prime) **(B-G)**

  (d arbitrary)

- $SL_3(p)$ **(B-G)** based on HELFGOTT

ARCHIMEDIAN SETTING

$\Lambda \subset SL_2(\mathbb{Z})$ non-elementary, finitely generated

$$B_N = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda |$$
$$\|\gamma\| = (a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}} \leq N\}$$

Theorem. (BGS)

There is a constant $C(\Lambda)$ *such that for*
$N \to \infty$

$$|\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B_N | r(abcd) < C(\Lambda)\}|$$

$$\gtrsim \frac{|B_N|}{(\log N)^4}$$

HYPERBOLIC LATTICE POINT COUNTING

Λ acting on $\mathbb{H} = \mathbb{H}^2 = \{x + iy \in \mathbb{C} | y > 0\}$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \quad gz = \frac{az + b}{cz + d}$$

$$\|g\|^2 = a^2 + b^2 + c^2 + d^2 = 4u(gi, i) + 2$$

$$\cosh d_H(z, w) = 1 + 2u(z, w)$$

$$u(z, w) = \frac{|z - w|^2}{4\text{Im}\, z\, \text{Im}\, w}$$

$L = L(\Lambda) \subset \mathbb{R} =$ limit set of Λ

$\delta = \delta(L) =$ Hausdorff dimension of $L \quad (0 < \delta \leq 1)$

$$|B_N = \{\gamma \in \Lambda | \ \|\gamma\| < N\}| \sim N^{2\delta}$$

$\delta > \frac{1}{2}$ LAX-PHILLIPS (wave equation methods)

$\delta \leq \frac{1}{2}$ LALLEY (methods from symbolic dynamics)

CASE $\delta(L) > \frac{1}{2}$

Spectrum of Laplace operator on $\Lambda\backslash\mathbb{H}$

$$0 \le \underset{\substack{\| \\ \delta(1-\delta)}}{\lambda_0(\Lambda)} < \lambda_1(\Lambda) \le \ldots \le \lambda_{\max}(\Lambda) < \frac{1}{4} \xrightarrow[\text{continuous}]{}$$

Theorem. (LAX-PHILLIPS)

$$\lambda_j = \delta_j(1-\delta_j) \qquad \delta_0 = \delta$$

$$|\{\gamma \in \Lambda | d_H(w, \gamma w_0) \le s\}| =$$

$$\sum_{j \ge 0} C_j \varphi_j(w) \varphi_j(w_0) e^{\delta_j s} + 0(e^{\frac{1}{3}(1+\delta_0)s})$$

Corollary.

$$|\{\gamma \in \Lambda| \ \|\gamma\| \le N\}| \sim N^{2\delta} + 0\big(N^{2\delta_1}\big)$$

SELBERG'S THEOREM

AND CONJECTURE

$$\Gamma(q) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\bmod q) \right\}$$

Theorem. (SELBERG) $\lambda_1\Big(\Gamma(q)\Big) \geq 3/16$

Conjecture. (SELBERG) $\lambda_1\Big(\Gamma(q)\Big) \geq \frac{1}{4}$

(no exceptional eigenvalues)

Theorem. (KIM-SARNAK) $\lambda_1\Big(\Gamma(q)\Big) > \frac{1}{4} - \left(\frac{7}{64}\right)^2$

GENERALIZATION OF SELBERG'S THEOREM

$$\Lambda = \langle S \rangle \subset SL_2(\mathbb{Z}) \quad \delta(\Lambda) > \frac{1}{2}$$

$$\Lambda_q = \{\gamma \in \Lambda : \gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{q}\}$$

$$\lambda_0(\Lambda_q) = \lambda_0(\Lambda)$$

Theorem. (BGS) $\lambda_1(\Lambda_q) > \lambda_0 + \varepsilon$
$\varepsilon = \varepsilon(\Lambda) > 0$ *and all square-free* $q \geq 1$

$L^2(\Lambda_q \backslash \mathbb{H}) \leftrightarrow H_q$ *equivariant functions on*

$$\Lambda \backslash \mathbb{H} \times SL_2(q)$$

Proof of spectral gap based on expansion of

$$\mathcal{G}(SL_2(q), \pi_q(S))$$

Corollary.

$q \in \mathbb{Z}_+, q$ *square-free*

$(q, q_0) = 1$

$g \in SL_2(q)$

$$|\{\gamma \in \Lambda| \ \|\gamma\| \leq N \text{ and } \pi_q(\gamma) = g\}|$$

$$\sim \frac{N^{2\delta}}{|SL_2(q)|} + 0(q^C N^{2\delta-\varepsilon})$$

with ε, C *depending on* Λ

GENERAL CASE

$\left(\text{no } L^2\text{-spectral theory for } \delta(\Lambda) \leq \frac{1}{2}\right)$

$\Lambda = \langle T_1, \ldots, T_k \rangle$ Schottky group with no parabolics

SYMBOLIC DYNAMICS

Λ $\qquad \sum_* =$ finite sequences on $\{\pm 1, \ldots, \pm k\}$ compatible with transition matrix

$$L(\Lambda) \qquad \sum \subset \prod_{n=0}^{\infty} \{\pm 1, \ldots, \pm k\}$$

$$L \xrightarrow{F} L \qquad \sum \xrightarrow{\sigma} \sum$$

NIELSEN map $\qquad$ finite type shrift

$f = \log |F'|$ $\qquad \tau(x) = d_H(i, xw) - d_H(i, x_2 x_3 \ldots w)$

distorsion function $\qquad x = x_1 x_2 x_3 \cdots \in \sum$

RENEWAL EQUATION

(LALLEY)

TRANSFER OPERATOR

$$S_n\tau = \tau + \tau\circ\sigma + \cdots + \tau\circ\sigma^{n-1}$$

$$S_n\tau(x) = d_H(i, x_1\cdots x_n x_{n+1}\ldots w) - d_H(i, x_{n+1}\cdots w)$$

$\psi : \sum \to \mathbb{R}$ Hölder

$$N(a,x) = \sum_{n\geq 0}\ \sum_{y:\sigma^n y = x} \psi(y)\ 1_{\{S_n\tau(y)\leq a\}}$$

$\psi = 1$

$$N(a,x) = |\{\gamma\in\Lambda | d_H(i,\gamma x w) - d_H(i, xw) \leq a\}|$$

Renewal Equation

$$N(a,x) = \sum_{\sigma x' = x} N(a - \tau(x'), x') + \psi(x)\ 1_{\{a\geq 0\}}$$

Laplace transform

$$F(z,x) = \int e^{az} N(a,x)da$$

satisfies

$$(I - \mathcal{L}_z)F(z,x) = \frac{\psi(x)}{z}$$

where $\mathcal{L}_z$ is the tansfer operator

$$(\mathcal{L}_z\varphi)(x) = \sum_{\sigma y = x} e^{z\tau(y)}\varphi(y)$$

Study of resolvent $(I - \mathcal{L}_z)^{-1}$

RUELLE'S THEOREM

$\mathcal{F} = \mathcal{F}_\rho(\sum)$ Hölder functions

$$\|\varphi\|_\rho = \sup_n \rho^{-n} \mathrm{var}_n(\varphi)$$

$\mathrm{var}_n(\varphi) = \sup |\varphi(x) - \varphi(y)| \quad x_j = y_j, 1 \le j \le n$

$f \in C\Big(\sum\Big)$ real valued

$$(\mathcal{L}_f \varphi)(\mathrm{x}) = \sum_{\sigma \mathrm{y} = \mathrm{x}} \mathrm{e}^{f(\mathrm{y})} \varphi(\mathrm{y})$$

Theorem. *$\mathcal{L}_f : \mathcal{F} \to \mathcal{F}$ has simple eigenvalue λ_f and eigenfunction $h_f > 0$.*

Remainder of spectrum $\subset \{|z| < \lambda_f - \varepsilon\}$

$\exists$ Borel probability measure ν_f on $\sum$

$$\mathcal{L}^* \nu = \lambda \nu \qquad \int h d\nu = 1$$

$$\lambda^{-n} \mathcal{L}^n g \to \Big(\int g d\nu\Big) h \qquad \forall g \in C\Big(\sum\Big)$$

LALLEY–NAUD

Reinterpretation of $\delta(\Lambda)$

$1 = \lambda_{-\delta f}$ $\qquad f =$ distorsion function

$\nu_{-\delta f} \sim \delta$-dim Hausdorff measure on $L(\Lambda)$.

(BOWEN, SERIES)

Theorem. (L-N)

$$\mathcal{L}_z : \mathcal{F} \to \mathcal{F}$$

- $(I - \mathcal{L}_z)^{-1}$ *meromorphic on* $Re\, z < -\delta + \varepsilon$ *with simple pole at* $z = -\delta$

- $\|(I - \mathcal{L}_z)^{-1}\| < C(1 + |Im\, z|^2)$ *for* $|z| \to \infty$

- $|\{\gamma \in \Lambda | d_H(i, \gamma(i)) \leq s\}| = Ce^{\delta s} + 0(e^{(\delta - \varepsilon)s})$

Corollary.

$$|\{\gamma \in \Lambda| \, \|\gamma\| \le N\}| \sim N^{2\delta} + 0(N^{2\delta-\varepsilon})$$

CONGRUENCE SUBGROUPS

Theorem. (BGS)

q square-free $\qquad (q, q_0) = 1$

$g \in SL_2(q)$

$$|\{\gamma \in \Lambda| \, \|\gamma\| \le N, \pi_q(\gamma) = g\}| \sim$$

$$\frac{N^{2\delta}}{|SL_2(q)|}\Big(1 + 0\Big(N^{-\frac{1}{\log\log N}}\Big)\Big) + q^C N^{2\delta-\varepsilon}$$

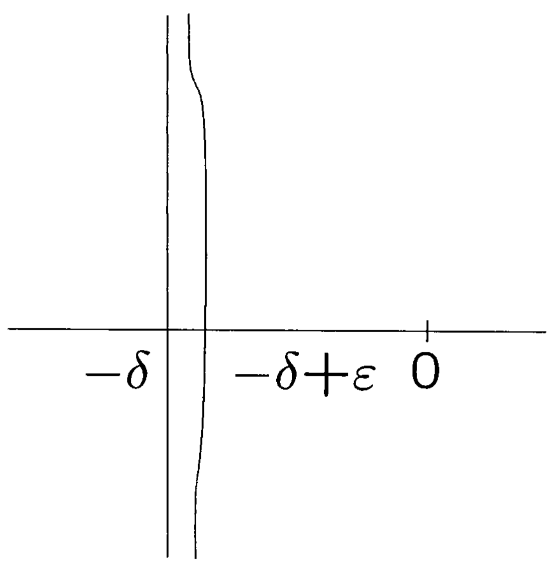
Extended action of $\mathcal{L}_z$ on $\mathcal{F}\big(\sum \times SL_2(q)\big)$

$$\ell^2\big(SL_2(q)\big) = \mathbb{R} \bigoplus_{q_1|q} E_{q_1}$$

Main estimate for $\mathcal{L}_z$ on $\mathcal{F}_{E_q}(\Sigma) = \mathcal{F}^{(q)}$

Proposition. $(I - \mathcal{L}_z)^{-1}\big|_{\mathcal{F}^{(q)}}$ holomorphic on

$$Rez < -\delta + \varepsilon \ \min\left\{1, \frac{\log q}{\log(1 + |\text{Im } z|)}\right\}$$



$$\|(I - \mathcal{L}_z)^{-1}\| < (q + |\text{Im } z|)^C$$

ROLE OF EXPANSION

Proposition.

$$\mu : SL_2(q) \to \mathbb{C}$$

satisfying

$$\|\pi_{q_1}[\mu]\|_\infty < q_1^{-\kappa}\|\mu\|_1 \qquad q_1|q$$

Then

$$\|\mu * \varphi\|_2 \le q^{-\kappa'}\|\mu\|_1 \ \|\varphi\|_2$$

for

$$\varphi \in E_q$$

CONJECTURE (BGS)

($SL_2(\mathbb{Z})$ analogue of Dirichlet's Theorem)

Λ non-elementary subgroup of $SL_2(\mathbb{Z})$

$b \in \mathbb{Z}^2$ primitive vector

$\mathcal{O} = \{gb | g \in \Lambda\}$

$\pi(\mathcal{O}) = \{x \in \mathcal{O} | x_1, x_2 \text{ are prime}\}$

Then

$$\pi(\mathcal{O}) \text{ is Zariski dense in } \mathbb{A}^2$$

if no local obstruction:

For every $q \geq 2$, there is $x \in \mathcal{O}$ such that

$$x_1 x_2 \in (\mathbb{Z}/q\mathbb{Z})^*$$

Theorem.

(conditional to Montgomery's conjecture)

$\Lambda \subset SL_2(\mathbb{Z}+i\mathbb{Z})$ *finitely generated*

Zariski dense in $SL_2(\mathbb{C}) \sim IS(\mathbb{H}^3)$

$$\delta\Big(L(\Lambda)\Big) > \frac{5}{3}$$

Then $|Im\, g_{1,2}|, g \in \Lambda$ *contains infinitely many primes, if no local obstruction*

Montgomery's Conjecture

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + 0(q^{-\frac{1}{2}}x^{\frac{1}{2}+\varepsilon})$$

SARNAK $\quad \exists\Lambda$ Schottky with $\delta(L) \geq 1,759\ldots$

DOYLE $\quad \delta\Big(L(\Lambda)\Big) < 2-\varepsilon_3$ for any geometrically finite Schottky Λ in $IS(H^3)$