Symmetric Homoclinic Bifurcation

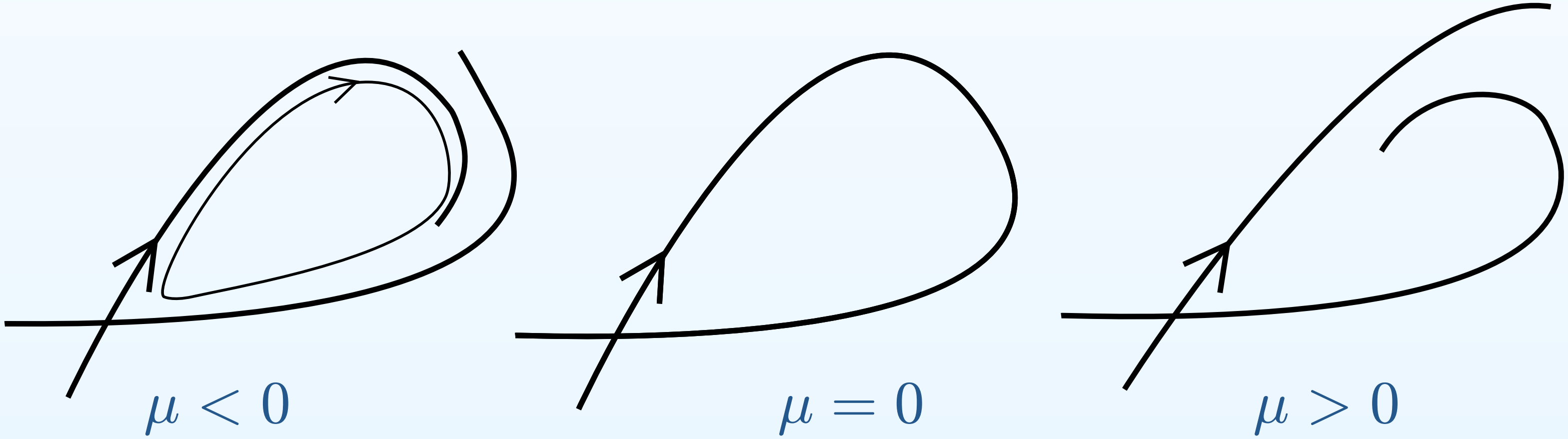
Alice Jukes
Imperial College, London, UK.

Work in collaboration with

- A.J. Homburg (Amsterdam),
- J. Knobloch (Ilmenau),
- J.S.W. Lamb (Imperial College).

Background

A codimension one homoclinic orbit to an equilibrium with real leading eigenvalues typically bifurcates in a blue sky catastrophe;



The set of non-wandering dynamics consists of a single periodic orbit for $\mu < 0$ and the empty set for $\mu \geq 0$.

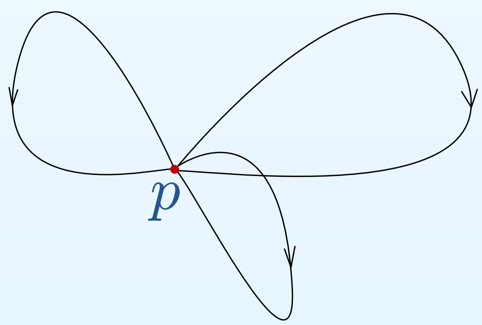
What happens when we add symmetry?

Setting

One parameter families of G-equivariant vector fields $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $f \in C^r$, $r \geq 2$,

$$\dot{x} = f(x, \mu)$$

for group G, with a hyperbolic equilibrium p.
Equivariance of f means that $x(t)$ is a solution $\Leftrightarrow$ $gx(t)$ is a solution, for all $g \in G$.



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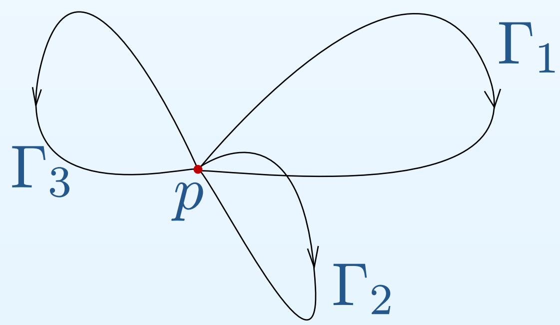


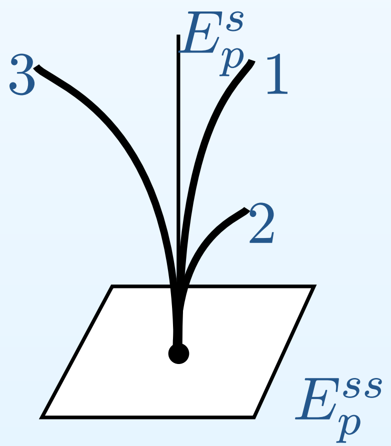
The isotropy of p is $G_p = \{g \in G | gp = p\}$.
The isotropy of Γ is $G_\Gamma = \{g \in G | g\gamma(t) = \gamma(t), \forall t \in \mathbb{R}, g \in G\}$.

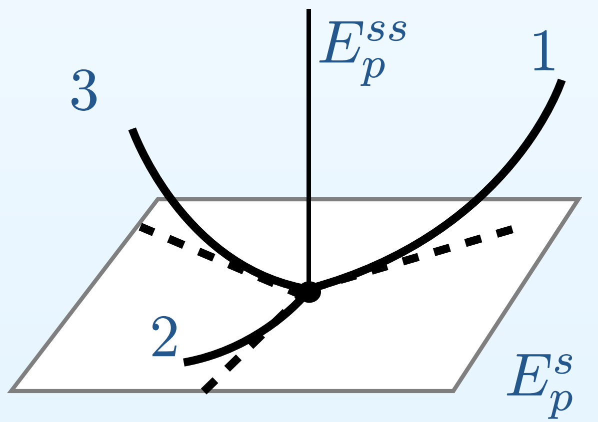
Relative homoclinic cycles

G_p acts on E_p^s and E_p^u as an irreducible representation.

Dihedral group of order 6, $\mathbb{D}_3(a, b)$, is the symmetry group of the triangle in the plane.
Subgroups $\mathbb{Z}_2(b)$, $\mathbb{Z}_2(ab)$ and $\mathbb{Z}_2(a^2b)$ and $\mathbb{Z}_3(a)$.

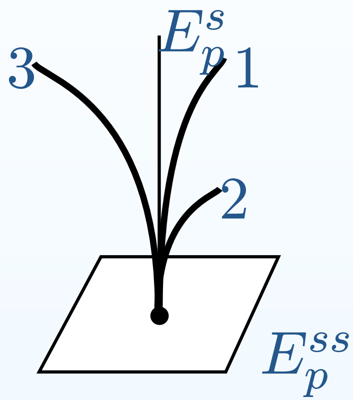
Example: $G_p = G = \mathbb{D}_3$, $G_\Gamma = \mathbb{Z}_2$

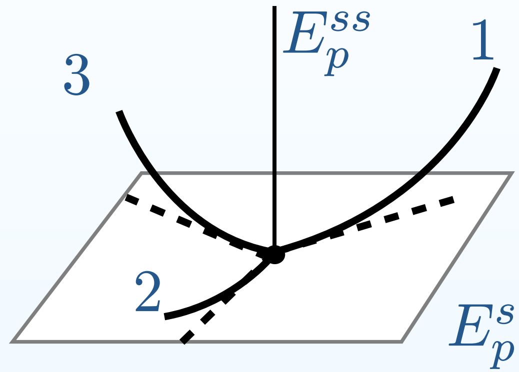




Trivial one dimensional representation

Non-trivial two dimensional representation





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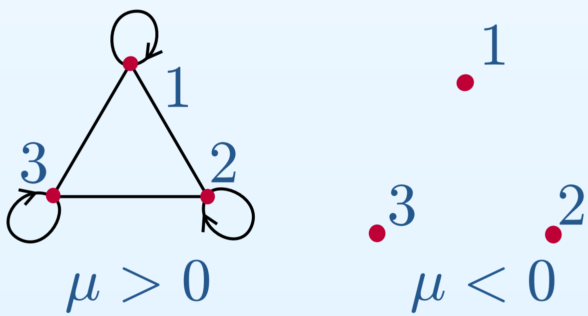
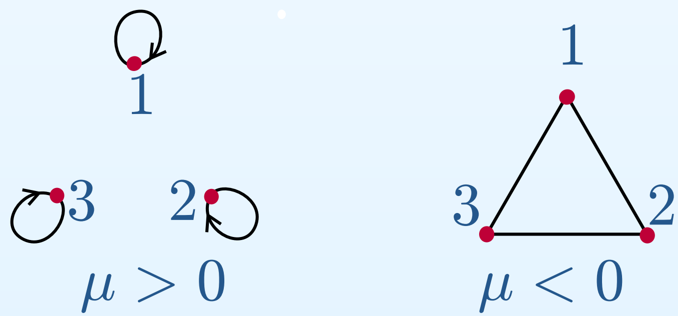




Idea of result

Given the following information about a relative homoclinic cycle:

- the leading stable and unstable eigenvalues at p, $-\rho^s$ and ρ^u, satisfy $\rho^s < \mathrm{Re}(\rho^u)$.
- G, symmetry group of cycle
- G_p, isotropy group of an equilibrium p
- h, element of G for which Γ connects p and hp
- G_Γ, isotropy group of Γ
- representation of G_p on E_p^s, the eigenspace in the direction of the leading stable eigenvalue

Then there is a construction giving a complete description of the non-wandering dynamics in terms of a symbolic system of $\frac{|G|}{|G_\Gamma|}$ symbols.

Shift dynamics

Label the connections in the relative homoclinic cycle, H, by $\Gamma_1, \ldots, \Gamma_k$.
A non-wandering orbit that follows a sequence of connections has an itinerary in $\mathcal{B}^k = \{1, ..., k\}^{\mathbb{Z}}$.

The connectivity matrix $C = [c_{ij}]$ for H is defined by

$$c_{ij} = \begin{cases} 1 & \omega(\gamma_i) = \alpha(\gamma_j) \\ 0 & \text{otherwise.} \end{cases}$$

Given matrix $M = [m_{ij}]$, topological Markov chain $\mathcal{B}^k_M$ is $\left\{\kappa \in \{1, ..., k\}^{\mathbb{Z}} : m_{\kappa_i \kappa_{i+1}} = 1\right\}$.

Main Theorem

Theorem 1 *(Modulo some technical assumptions)*
Assume that the eigenvalue of p with smallest real part is real and negative and that at $\mu = 0$ there exists a codimension one relative homoclinic cycle H.

Then there are matrices M_+ and M_- giving a symbolic description of the non-wandering dynamics, $\mathcal{B}^k_{M_+}$ and $\mathcal{B}^k_{M_-}$.

These descriptions are topologically conjugate to the set of non-wandering orbits for cycle H for $\mu > 0$ and $\mu < 0$ respectively if and only if $M_+ + M_- = C$.

Idea of proof

Lin's method, developed by J. Hale, X.-B. Lin and B. Sandstede gives bifurcation equations.

In fact, we only need to show the existence of solutions to a set of truncated equations: for an orbit to follow $\Gamma_{\kappa_{i-1}}$ then Γ_{κ_i} there is one dimensional equation,

$$0 = \mu + c^s(\kappa_{i-1}, \kappa_i)\mathrm{e}^{-2\rho^s \omega_i}$$

where ω_i is a transition time for this section of orbit.
Conjugacy by uniqueness from Lin's method.

The important thing in solving these equations is the sign of the constant $c^s(\kappa_{i-1}, \kappa_i)$.

Constants in the bifurcation equations

Now $c^s(i,j)$ is an inner product of

$$e_i^s := \lim_{t\to\infty} \frac{\gamma_i(t)}{\|\gamma_i(t)\|} \in E^s_{\omega(\gamma_i)}$$

and

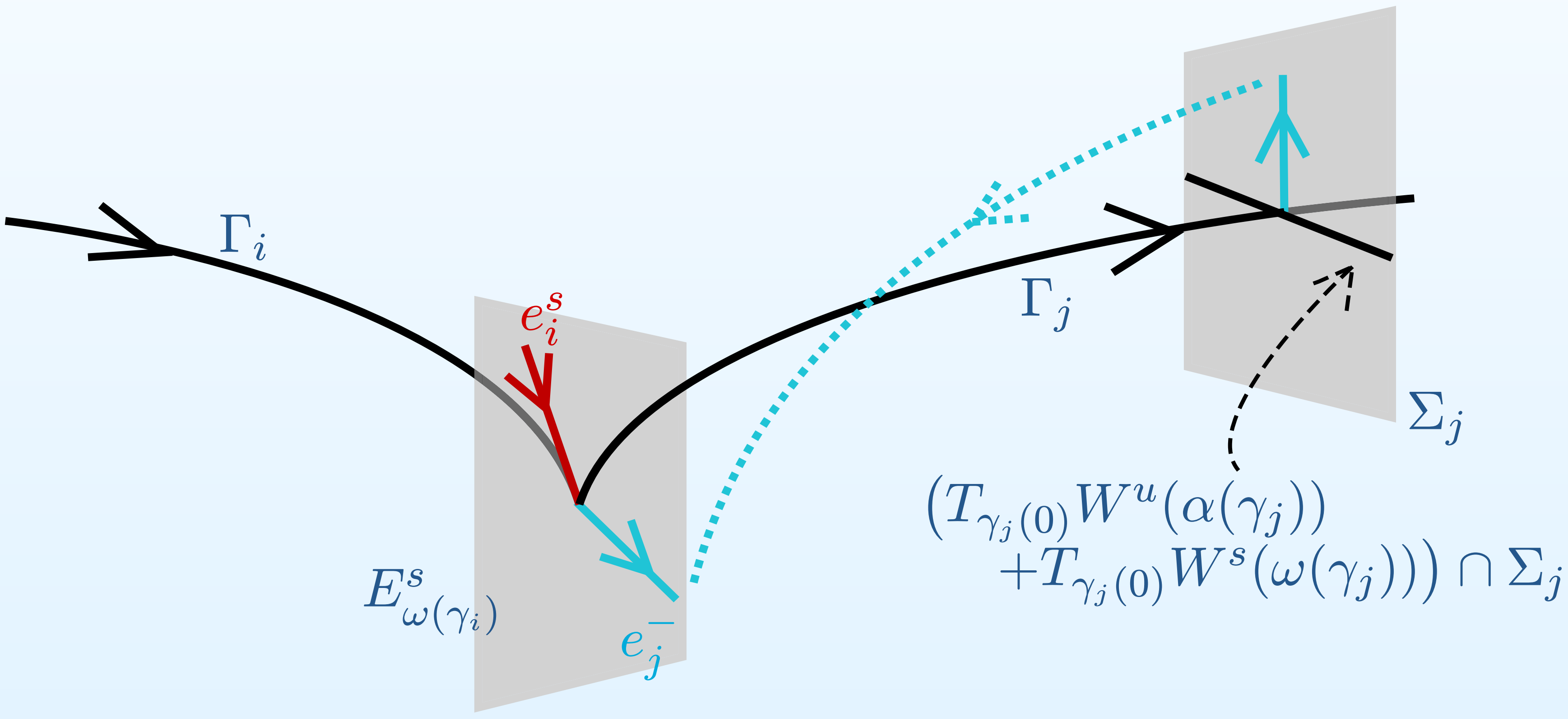
$$e_j^- := \lim_{t\to-\infty} \frac{\delta_j(t)}{\|\delta_j(t)\|} \in E^s_{\alpha(\gamma_j)}$$

where $\delta_j(t)$ is the solution to the adjoint variational equation

$$\dot{v}(t) = -D_1 f(\gamma_j(t), 0)^* v(t).$$

Geometric interpretation of the constants

In fact, e_j^- is equal to the flow back of a vector which is a basis of the space $(T_{\gamma_j(0)}W^u(\alpha(\gamma_j)) + T_{\gamma_j(0)}W^s(\omega(\gamma_j)))^\perp$ from Σ_j by the adjoint of the flow linearised around the connection.



The matrices M_+ and M_- are defined by the following;
$A := (a_{ij})_{i,j\in\{1,\dots,k\}}$ has entries

$$a_{ij} = \begin{cases} 0, & \text{if } \omega(\gamma_i) \neq \alpha(\gamma_j), \\ -\text{sign}\langle e_i^s, e_j^- \rangle, & \text{if } \omega(\gamma_i) = \alpha(\gamma_j). \end{cases}$$

Write

$$M_+ = 1/2(A + |A|), \quad M_- = -1/2(A - |A|).$$

Symmetry relations of $c^s(i, j)$

The symmetry of the vector field transfers to symmetry of the constants, we have

Lemma 2 *For all* $g \in G$

$$g e_i^s = e_{g(i)}^s \in E_{g(i)}^s \text{ and } g e_j^- = e_{g(j)}^- \in E_{g(j)}^s.$$

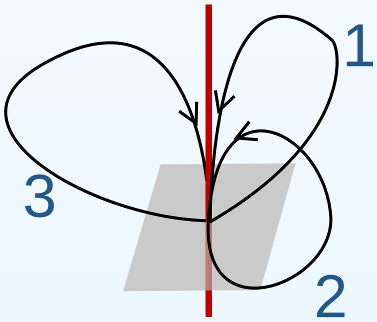
Thus, for all $g \in G$,

$$g c^s(i, j) = c^s(g(i), g(j)). \tag{1}$$

Example

$G_p = G$, representation on E^s_p is one dimensional and trivial with $G_\Gamma = \mathbb{Z}_2$.

Homoclinic cycle $\Rightarrow C$ is a 3×3 matrix with every entry 1.

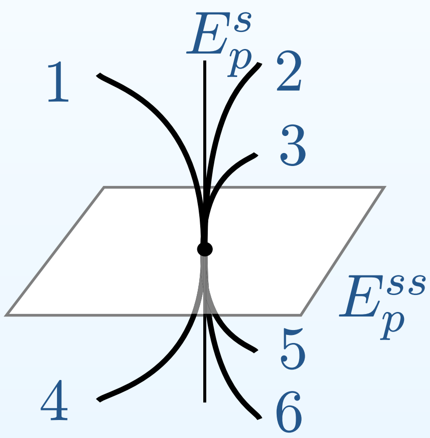


All orbits approach from the same direction:
e^s_i same $\forall\ i$.
e^-_j same $\forall\ i$.

Then up to a change of sign there is a full shift on three symbols for $\mu > 0$ and no non wandering dynamics for $\mu < 0$.

Example

$G_p = G$, representation on E^s is one dimensional, non-trivial with $G_\Gamma = \{id\}$.



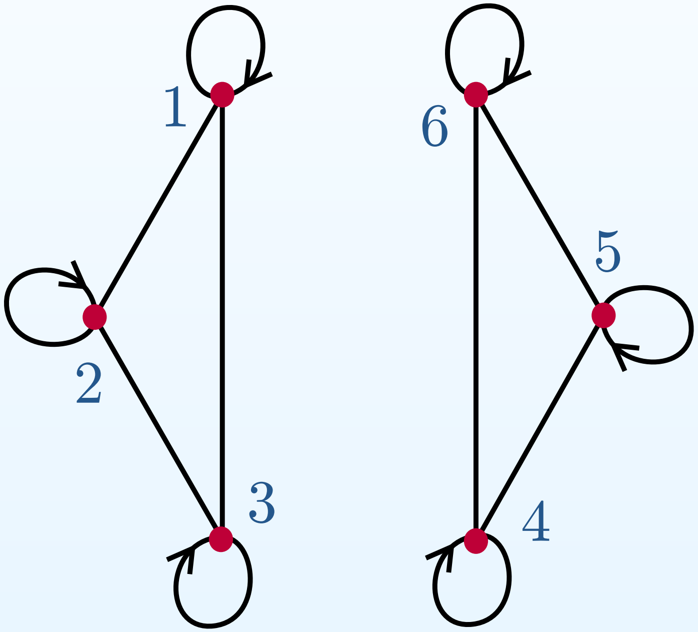
sign$e^s_i \neq$ signe^s_j for $i \in \{1,2,3\}$, $j \in \{4,5,6\}$.

First column of A is inner product of these with e^-_1.

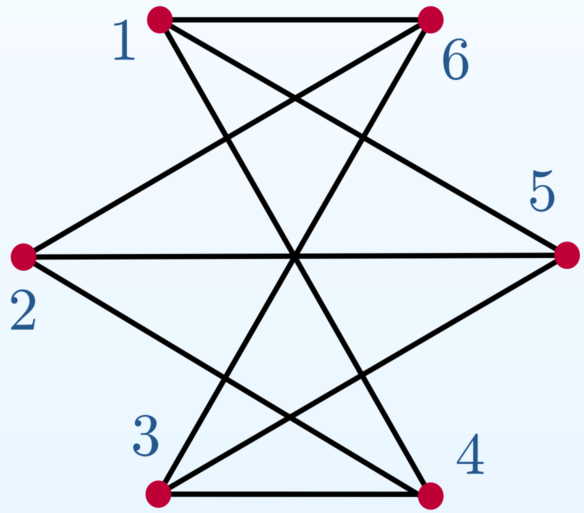
Thus, using equation (1), up to a change of sign,

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Markov graphs



$\mu > 0$

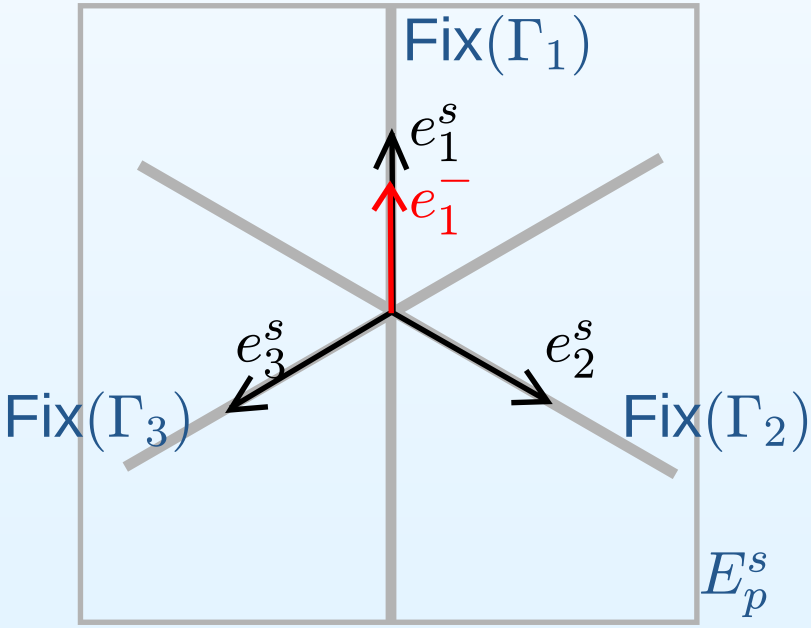


$\mu < 0$

Example

$G_p = G$, representation on E_p^s is two dimensional, non-trivial with $G_\Gamma = \mathbb{Z}_2$

Representation on E_p^s is as $\mathbb{D}_3 \Rightarrow \exists$ 3 one dimensional fixed point spaces of order two elements of G. Each of e_i^s and e_i^- must be contained in one of these since G_{Γ_i} contains these symmetries.



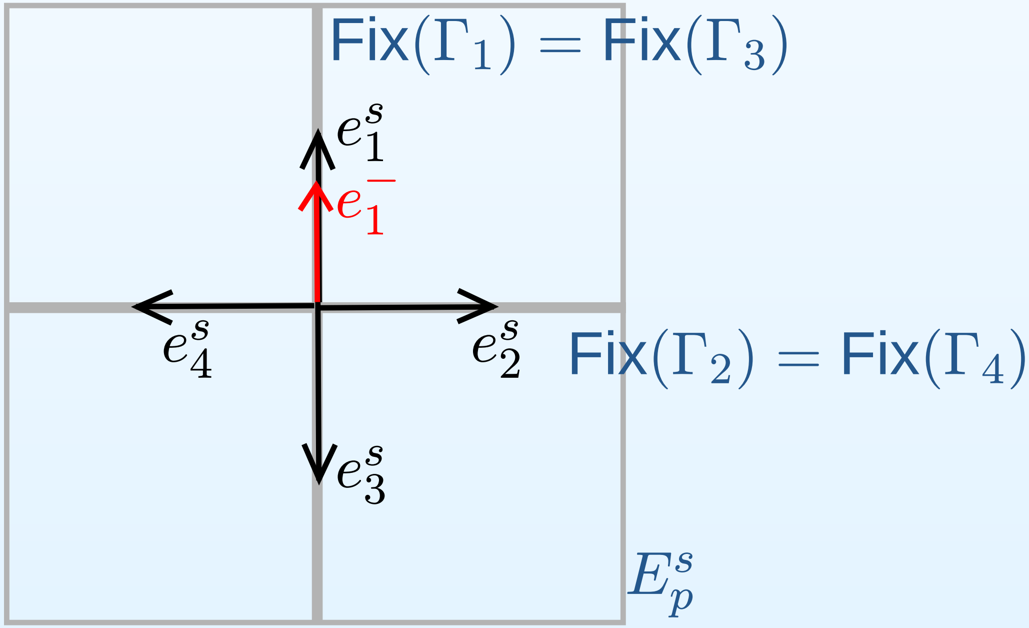
$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Example: When $M_+ + M_- \neq C$

$G_p = G = \mathbb{D}_4$, representation on E^s is two dimensional, non-trivial with $G_\Gamma = \mathbb{Z}_2$

Representation on E^s_p is as $\mathbb{D}_4 \Rightarrow \exists$ 2 one dimensional fixed point spaces of order two elements of G.

The leading order terms in the bifurcation equations are zero!



General setting

To ensure the connection is of codimension one, we assume that inside Fix(G_Γ) the connection is of codimension one,

$$\dim\left(W^u(p)\cap \mathsf{Fix}(G_\Gamma)\right) = \dim\left(W^u(hp)\cap \mathsf{Fix}(G_\Gamma)\right).$$

We only need to consider connected cycles since we are interested in non-wandering orbits:

Lemma 3 *A codimension one relative homoclinic cycle with transitive group action is connected* $\Leftrightarrow G = \langle h, G_p\rangle$ *where* Γ *is a connection between* p *and* hp*.*

Assumptions

Assumption	Reason
G_p acts absolutely irreducibly on E_p^s	$\rho^s \in \mathbb{R}$
Γ is non-degenerate, $1 = \dim(T_{\gamma(0)} W^u(hp) \cap T_{\gamma(0)} W^s(p))$	To avoid higher dimensional intersections of the manifolds
Γ approaches along the leading stable direction, $\lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|} \in E_p^s$	Not always the case is generic symmetric systems - excludes orbit flip.
Fix$(G_\Gamma) \cap E_p^s \neq \{0\}$	To avoid inclination flip type situation

Main Theorem

Suppose that G is a compact Lie group.
For all i let Σ_i be a transversal section to Γ_i at $\gamma_i(0)$ such that $g\Sigma_i = \Sigma_{g(i)}$.
Let Π_μ be the first return map to the union $\cup_{j=1}^k \Sigma_j$ and let $\mathcal{D}_\mu$ be an invariant set for Π_μ.

Then with the assumptions above, there exists matrices M_+ and M_- such that, when when $M_+ + M_- = C$, for each $\kappa \in \mathcal{B}^k_{M_+}$ there is a unique $x \in \mathcal{D}_\mu$ with $\Pi^i_\mu(x) \in \Sigma_{\kappa_i}$ for $\mu > 0$. Similarly for $\mu < 0$ with M_+ replaced by M_-.

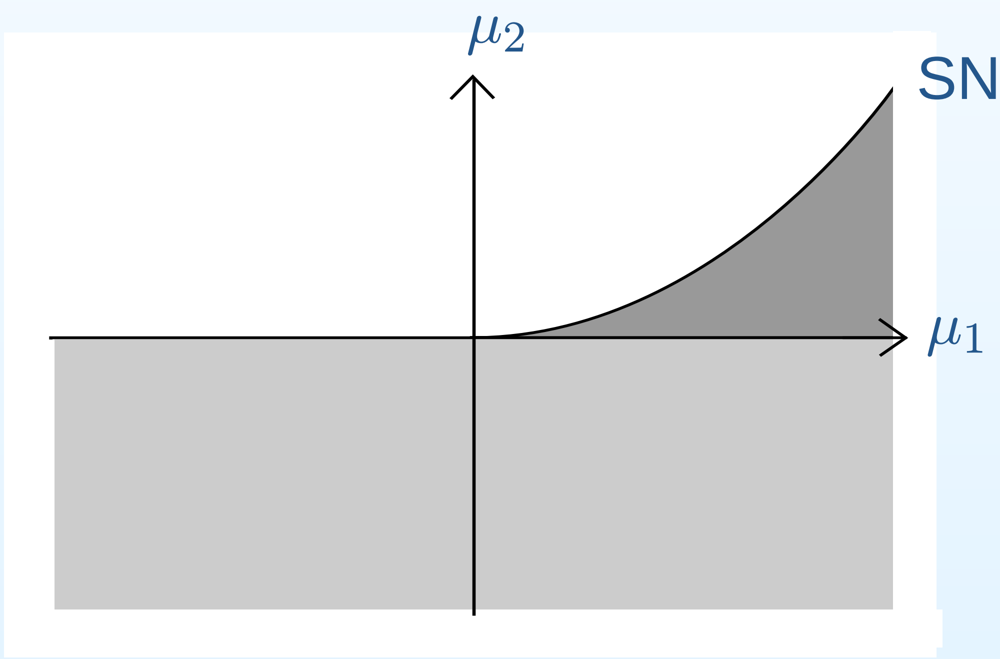
Related Results

- Codimension one relative homoclinic cycles where the leading eigenvalue is complex:
  There are many orbits for each itinerary, infinitely many when $\mu = 0$.
- Codimension two relative homoclinic cycles with symmetry:
  Codimension two resonant homoclinic loop with $\mathbb{D}_3$ symmetry - complicated structure of bifurcations.
- Systems with time-reversing symmetries:
  Non-wandering dynamics near robust reversible heteroclinic cycles also admits a symbolic description.

Preprint: Bifurcation from codimension one relative homoclinic cycles, A.J. Homburg, A.C. Jukes, J. Knobloch, J.S.W. Lamb.
Thesis: On Homoclinic Bifurcation with Symmetry, 2006, Imperial College, London.

Codimension two resonant homoclinic loop

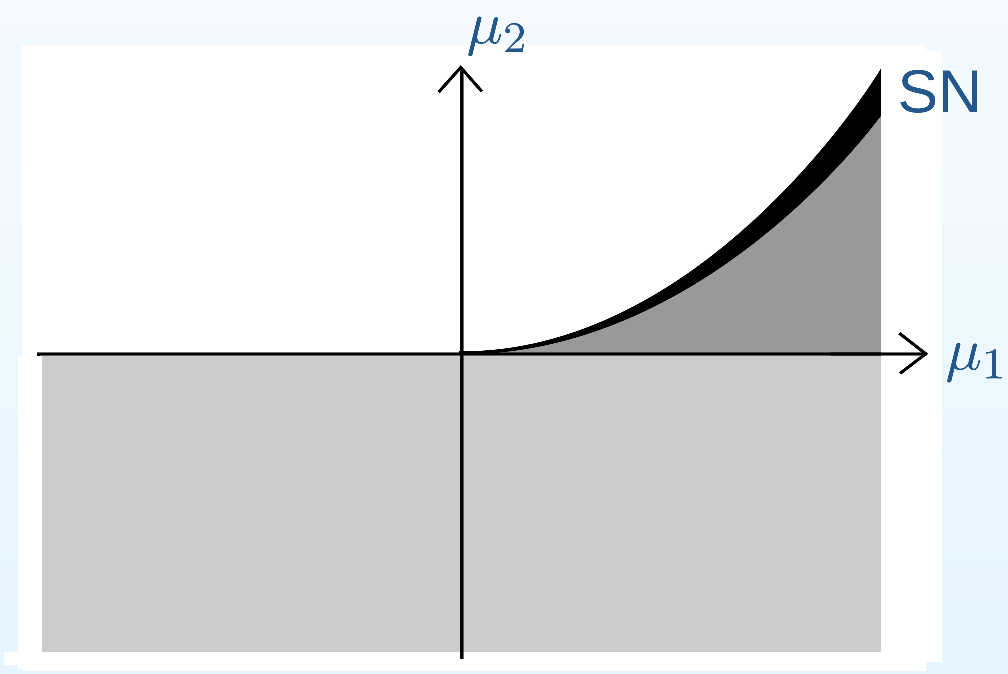
Assume the leading eigenvalues of p are real and the ratio is controlled by μ_1 such that they are resonant at $\mu_1 = 0$.
Assume that at $\mu_2 = 0$ there is a non-twisted non-degenerate homoclinic orbit to p
The bifurcation diagram near $\mu_1 = \mu_2 = 0$ is;



Codimension two resonant homoclinic cycle

$G = \mathbb{D}_3$, $G_p = G$ and $G_\Gamma = \mathbb{Z}_2$.

Case 1 Representations on E_p^s and E_p^u one-dimensional and trivial.



Case 2 Representations on E_p^s one-dimensional and trivial, and on E_p^u two-dimensional and faithful.