# Spatial and temporal feedback control of traveling wave solutions of the two-dimensional complex Ginzburg–Landau equation.

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# Outline

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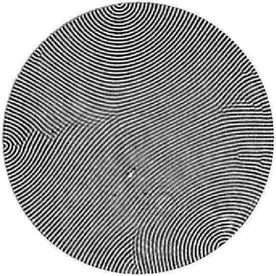
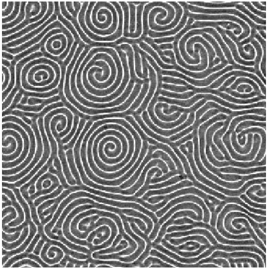
# Patterns



Y.-C. Hu, R. Ecke, and G. Ahlers, Phys.

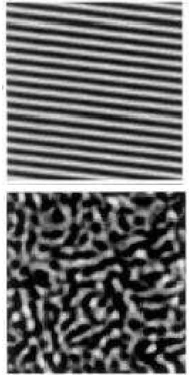
Rev. E 51, 3263 (1995)

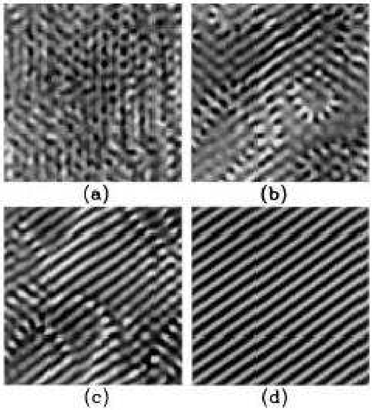


J. Liu, K.M.S. Bajaj, and G. Ahlers, unpublished

# Traveling waves

Lu, Yu and Harrison (1996), control of patterns in a nonlinear optics problem.





# Feedback control

Method of Pyragas (1992)

- Suppose a system

$$\dot{A} = f(A), \qquad A = A(t)$$

  contains a number of unstable periodic orbits.
- Feedback is of the form

$$F = A(t) - A(t - T)$$

  where $T$ is the period of the targeted orbit.
- $\Rightarrow$ feedback is non-invasive, and only information required about the unstable orbit is it's period.
- This method has been used successfully in laboratory experiments on electronic, laser and chemical systems.

# Outline

# The Complex Ginzburg–Landau Equation

- Amplitude equation for oscillatory instability (Hopf bifurcation) in a spatially-extended system.
- Ansatz $U = A(\mathbf{x}, t)\mathrm{e}^{i\omega_H \tau}$, then $A$ satisfies

$$\frac{\partial A}{\partial t} = A + (1 + ib_1)\nabla^2 A - (b_3 - i)|A|^2 A,$$

- Exact traveling wave solutions:

$$A_{\mathrm{TW}} = R\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x} + i\omega t},$$

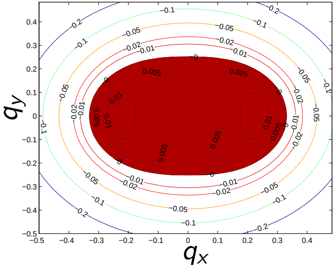
$$R^2 = \frac{1 - k^2}{b_3}, \qquad \omega = R^2 - b_1 k^2, \qquad k = |\mathbf{k}|,$$

- Unstable to longwave perturbations if $b_1 > b_3 > 0$, $\Rightarrow k < 1$.

# The CGLE with no feedback

- The Benjamin–Feir instability. If $b_1 > b_3 > 0$, all longwave perturbations ($q \ll 1$) grow.
- Perturbations with $\mathbf{k} \cdot \mathbf{q} = 0$ grow if

$$q^2 < q_{\mathrm{cr}}^2 = \frac{2R^2(b_1 - b_3)}{1 + b_1^2}.$$



# Spatial and temporal feedback

- Feedback is added in order to stabilize the waves:

$$\frac{\partial A}{\partial t} = A + (1 + ib_1)\nabla^2 A - (b_3 - i)|A|^2 A + F,$$

- where

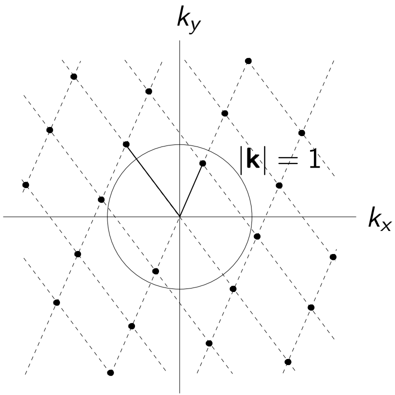
$$F = \gamma[A(\mathbf{x}, t) - A(\mathbf{x}, t - \Delta t)] + \sum_{j=1}^{N} \rho_j [A(\mathbf{x} + \Delta\mathbf{x}_j, t) - A(\mathbf{x}, t)].$$

- we choose

$$\Delta t|\omega| = 2\pi \qquad \Delta\mathbf{x}_j \cdot \mathbf{k} = 2\pi n_j, \quad n_j \in \mathbb{Z}.$$

  so that the feedback is non-invasive.
- Consider $N = 1$ and $N = 2$.

# $N = 2$




- For given $\Delta\mathbf{x}_1, \Delta\mathbf{x}_2$, there is a 2D lattice of wavevectors $\mathbf{k}$ which satisfy $\Delta\mathbf{x}_j \cdot \mathbf{k} = 2\pi n_j$.
- $\Rightarrow$ can choose $\Delta\mathbf{x}_1, \Delta\mathbf{x}_2$ to target a particular $\mathbf{k}$.

# Stability analysis

- Consider small amplitude perturbations, with wavenumber $\mathbf{q}$, $|\mathbf{q}| = q$.

$$A = R\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}+i\omega t}(1 + a_+(t)\mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}} + a_-(t)\mathrm{e}^{-i\mathbf{q}\cdot\mathbf{x}}),$$

- Linearize:

$$\frac{d}{dt}\begin{pmatrix} a_+(t) \\ a_-^*(t) \end{pmatrix} = J\begin{pmatrix} a_+(t) \\ a_-^*(t) \end{pmatrix} + \gamma\left[\begin{pmatrix} a_+(t) \\ a_-^*(t) \end{pmatrix} - \begin{pmatrix} a_+(t-\Delta t) \\ a_-^*(t-\Delta t) \end{pmatrix}\right],$$

- where

$$J = \begin{pmatrix} -c_1 q^2 - c_2 R^2 & -c_2 R^2 \\ -c_2^* R^2 & -c_1^* q^2 - c_2^* R^2 \end{pmatrix} + 2\mathbf{k}\cdot\mathbf{q}\begin{pmatrix} -c_1 & 0 \\ 0 & c_1^* \end{pmatrix}$$
$$+ \sum_{i=1}^{N} \rho_j(\mathrm{e}^{i\mathbf{q}\cdot\Delta\mathbf{x}_j} - 1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

# Outline

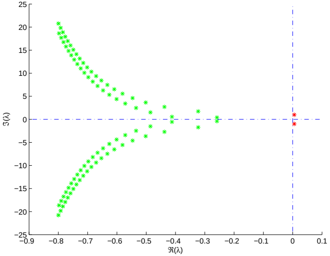
# Delay equations

$$\dot{z}(t) = (\alpha + i\beta)z(t) - \gamma(z(t) - z(t - \Delta t))$$

- Write $z = \mathrm{e}^{\lambda t/\Delta t}$, then

$$\lambda = \Delta t\left((\alpha + i\beta) - \gamma(1 - \mathrm{e}^{-\lambda})\right)$$

- This equation has an infinite number of solutions $\lambda \in \mathbb{C}$.
- If any of them have a positive real part, then the $z = 0$ solution is unstable.



# Nakajima (1997)

A result on the failure of feedback to stabilize periodic orbits:

- Consider the ODE

$$\dot{x} = f(x(t), t), \qquad x \in \mathbb{R}^n,$$

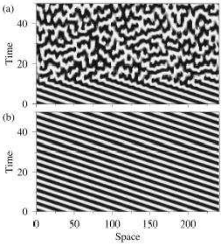
- $x^*(t)$ is an unstable periodic orbit with period $T$
- Add feedback:

$$\dot{x}(t) = f(x(t), t) + K(x(t) - x(t - T)),$$

Theorem: If the linearization about the orbit $x^*(t)$ has an odd number of positive, real unstable Floquet multipliers, then there is no value of $K$ for which $x^*(t)$ is a stable periodic solution.

Bleich and Socolar (1996)

- Use Pyragus-type feedback to stabilize TW in the 1D CGLE:




Montgomery and Silber (2004)

- Use combination of spatial and temporal feedback to stabilize TW in the 1D CGLE.

Harrington and Socolar (2001)

- For the 2D CGLE, stabilization of plane waves cannot be achieved with temporal feedback alone (i.e. $\rho_j = 0$), due to result of Nakajima.

$\Rightarrow$ what about 2D CGLE with spatial and temporal feedback?

# Perturbations unaffected by spatial feedback

$$J = \begin{pmatrix} -c_1 q^2 - c_2 R^2 & -c_2 R^2 \\ -c_2^* R^2 & -c_1^* q^2 - c_2^* R^2 \end{pmatrix} + 2\mathbf{k}\cdot\mathbf{q} \begin{pmatrix} -c_1 & 0 \\ 0 & c_1^* \end{pmatrix} + \sum_{i=1}^{N} \rho_j (\mathrm{e}^{i\mathbf{q}\cdot\Delta\mathbf{x}_j} - 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- If $\mathbf{q}\cdot\Delta\mathbf{x}_j = 2\pi n_j$, for some $n_j \in \mathbb{Z}$, then $J$ is the same as if $\rho_j = 0$.
- Hence, if there exists any $\mathbf{q}$ which satisfies

$$\mathbf{k}\cdot\mathbf{q} = 0, \quad \mathbf{q}\cdot\Delta\mathbf{x}_j = 2\pi n_j, \quad n_j \in \mathbb{Z} \quad \text{and} \quad |\mathbf{q}| < q_{cr},$$

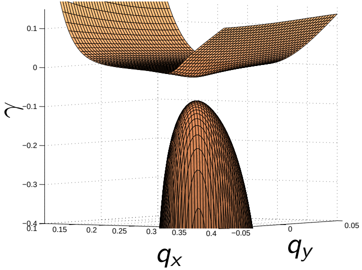
then the traveling waves cannot be stabilized.

# Outline

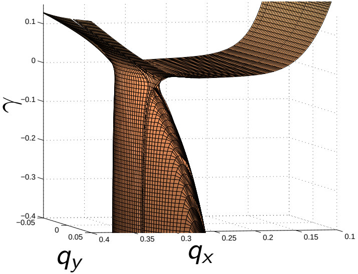
# Numerical results

- We use the Matlab package DDE-BIFTOOL to analyze the linearized system.
- Find surfaces of Hopf bifurcations in $\mathbf{q} - \gamma$ space.
- Compute the Floquet exponents for all $\mathbf{q}$ with other parameter fixed.

# Surfaces of Hopf bifurcations



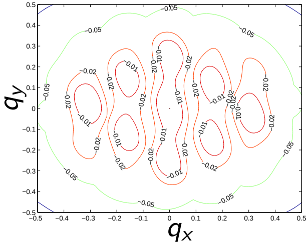

Stable



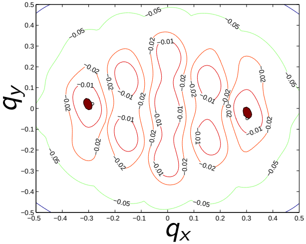

Unstable

- Montgomery and Silber find that $\gamma = -\frac{1}{\Delta t}$ is optimal choice.

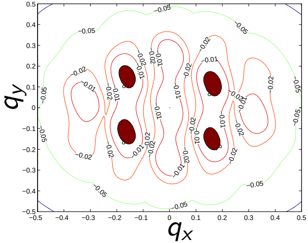
# Floquet exponents



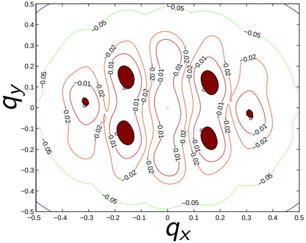

$|\mathbf{k}| = 0.285, N = 2$




$|\mathbf{k}| = 0.287, N = 2$




$|\mathbf{k}| = 0.283,\ N = 1$




$|\mathbf{k}| = 0.285,\ N = 1$

# Summary

- Many pattern forming systems contain patterns which are unstable.
- We use a particular form of non-invasive feedback to stabilize traveling waves in the complex Ginzburg–Landau equation.
- In two spatial dimensions, temporal feedback is not enough, but the waves can be stabilized with additional spatial feedback.