Traveling waves in porous media combustion:

Uniqueness of waves for small thermal diffusivity

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Combustion in Porous Medium (CPM)

Combustible gas or gas mixture:

oxygen, methane-air, hydrogen-air, propane-air...

Porous medium:

coal, ceramic fiber felt, polyurethane foam...

Applications:

power engineering, chemical and building technology,

ecology, fire and explosion safety

Close relatives:

convective burning occurring in combustion of granular explosives,

combustion in thin rough tubes

Mathematical Model for CPM

Energy $$c_p\rho(\Theta_\tau + u\Theta_\xi) - (\Pi_\tau + u\Pi_\xi) = qW + (c_p\rho D_{th}\Theta_\xi)_\xi$$

Concentration $$\rho(C_\tau + uC_\xi) = -W + (\Theta^{-1}D_{mol}(\rho\Theta C)_\xi)_\xi$$

Chemical kinetics $$W = Z_\rho C exp(-E/R\Theta)$$

Continuity $$\rho_\tau + (\rho u)_\xi = 0$$

Momentum $$\rho u = -K\nu^{-1}\Pi_\xi$$

State $$\rho = P/(c_p - c_\nu)\Theta$$

u - gas velocity, C - concentration of the deficient reactant, ρ, Π, Θ - density, pressure, temperature of the gas-solid system, W - chemical reaction rate, ν - kinematic viscosity, Z -frequency factor, E - activation energy, R - universal gas constant, q - heat release, c_p /c_v/ - specific heat at constant pressure /volume/, D_{th} /D_{mol}/ - thermal /molecular/ diffusivity

Derivation of Simplified Model

- **Small heat release approximation: variation of pressure, temperature, density and gas velocity assumed small**

  nonlinear effects are ignored everywhere except in the reaction term

- **Scaling:**

$$T = \frac{\Theta - \Theta_0}{\Theta_\infty - \Theta_0}, \quad P = \frac{\Pi - \Pi_0}{\Pi_\infty - \Pi_0}, \quad Y = \frac{C}{C_0}$$

Θ_0, Π_0, C_0 **-temperature, pressure, concentration at** $\tau = 0$,

$\Theta_\infty, \Pi_\infty$ **at** $\tau \to \infty$ **in case of homogeneous explosion**

- $t = \frac{\tau}{\bar{\tau}}$, $x = \frac{\xi}{\bar{\xi}}$, **where** $\bar{\tau}$, $\bar{\xi} = const$

Model of subsonic detonation [Sivashinsky 2002]

Propagation of the combustion fronts in highly resistable media:

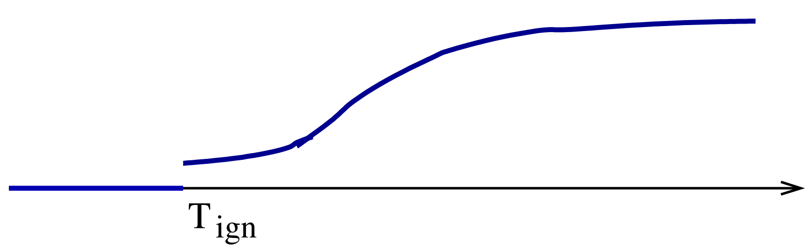
$$
\begin{aligned}
T_t - (1-\gamma^{-1})P_t &= \varepsilon T_{xx} + Y\Omega(T) \\
P_t - T_t &= P_{xx} \\
Y_t &= \varepsilon \mathrm{Le}^{-1} Y_{xx} - \gamma Y \Omega(T)
\end{aligned}
$$

$\varepsilon \ll 1$ - thermal diffusivity / pressure diffusivity ($\sim 10^{-4} - 10^{-7}$)

$\gamma > 1$ - specific heat ratio

$\mathrm{Le} = 1$ - Lewis number = thermal diffusivity / molecular diffusivity

$\Omega(T)$:
- of the Arrhenius type with an ignition cut-off at $T = T_{ign}$
- increasing on $(T_{ign}, +\infty)$
- Lipschitz continuous everywhere except $T = T_{ign}$



Different modes of combustion

At small ε, there are two distinct modes of combustion:

- **Deflagration**

  slow wave sustained by the diffusive transfer of heat

  $\Pi \sim \sqrt{\varepsilon}$ **and velocity of propagation:** $\sqrt{\varepsilon}$

  no traveling wave solutions

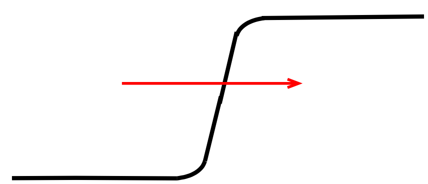
- **Subsonic Detonation**

  fast wave sustained by the diffusive transfer of pressure

  velocity of propagation: $O(1)$

  traveling waves exist

Fronts connecting the completely burnt state to the unburnt state



Subsonic Detonation: existence and uniqueness

Traveling wave ODE ($\xi = x - ct$, **c -free parameter)**

$$
\begin{aligned}
-cT' + c(1-\gamma^{-1})P' &= \varepsilon T'' + Y\Omega(T) \\
P'' &= c(T' - P') \\
cY' + \varepsilon Y'' &= \gamma Y \Omega(T)
\end{aligned}
$$

Boundary Conditions at $-\infty$: $P = 1,\ T = 1,\ Y = 0$

$+\infty$: $P = 0,\ T = 0,\ Y = 1$

- $\varepsilon = 0$: **There exists a unique** c **such that a front exists.** [**Gordon, Kamin, Sivashinsky 2002**]
- $\varepsilon > 0$: **Solutions of the system with** $\varepsilon > 0$ **exist and converge to the solution of the system with** $\varepsilon = 0$ **as** $\varepsilon \to 0$. **[Gordon, Ryzhik]**
- $\varepsilon > 0$: **The front is unique for small** $\varepsilon > 0$. [**G., Gordon, Jones**]

Result & Methods

Theorem. For sufficiently small $\varepsilon > 0$, there is a unique value of $c = c(\varepsilon)$ for which the system has an orbit satisfying boundary conditions:

$$(Q, P, T, Y) \to (\gamma^{-1}, 1, 1, 0) \text{ at } -\infty$$

$$(Q, P, T, Y) \to (0, 0, 0, 1) \text{ at } +\infty$$

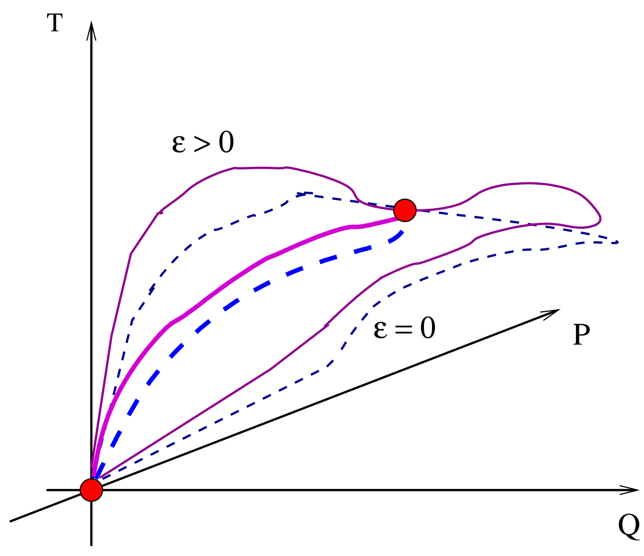
The orbit is unique, and hence the traveling wave up to translation.

$$\begin{aligned} \varepsilon T'' &= -cT' + c(1-\gamma^{-1})P' - Y\Omega(T) \\ P'' &= c(T' - P') \\ \varepsilon Y'' &= -cY' + \gamma Y \Omega(T) \end{aligned}$$

- System $[\varepsilon > 0]$ is a singular perturbation of System $[\varepsilon = 0]$
- Fenichel's invariant manifold theory is applicable

Strategy

- **Construct a manifold M_0 where the front of the system [$\varepsilon = 0$] lives**
- **Show that M_0 for small $\varepsilon > 0$ perturbs to a unique invariant manifold M_ε of the perturbed system**
- **Show that close M_ε no traveling wave can exist off M_ε**
- **Reduce the dimensions by restricting the flow to M_ε**
- **Extend the information about the front on M_0 to M_ε**



Slow System and Fast System

Introduce: $Q = \frac{1}{c}\int_{\xi}^{+\infty} Y\Omega(T)\,\mathrm{d}x$

Slow system : $' = \frac{\partial}{\partial \xi}$

$$\begin{aligned} Q' &= -c^{-1}Y\Omega(T) \\ P' &= c(T-P) \\ \varepsilon T' &= c(1-\gamma^{-1})P - cT + cQ \\ \varepsilon Y' &= c(1-Y) - \gamma c Q \end{aligned}$$

Fast system: $\eta = \frac{1}{\varepsilon}\xi, \quad \dot{} = \frac{\partial}{\partial \eta}$

$$\begin{aligned} \dot{Q} &= -\varepsilon c^{-1}Y\Omega(T) \\ \dot{P} &= \varepsilon c(T-P) \\ \dot{T} &= c(1-\gamma^{-1})P - cT + cQ \\ \dot{Y} &= c(1-Y) - \gamma c Q \end{aligned}$$

Critical manifold M_0

Slow System, $\varepsilon = 0$:

$$M_0 \qquad T = (1-\gamma^{-1})P + Q$$
$$Y = 1 - \gamma Q$$

Flow on M_0:

$$Q' = c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P),$$
$$P' = -c\gamma^{-1}P + cQ$$

- M_0 is **normally hyperbolic** and **attracting**

Assume that $\Omega(T)$ is smooth

Fenichel's Theory $\Longrightarrow$ for sufficiently small ε critical manifold M_0 perturbs to an invariant manifold M_ε: $O(\varepsilon)$ far from M_0

Slow Manifold M_ε

$$M_\varepsilon \qquad \begin{aligned} T &= (1-\gamma^{-1})P + Q + O(\varepsilon) \\ Y &= 1 - \gamma Q + O(\varepsilon) \end{aligned}$$

The flow on M_ε:

$$\begin{aligned} Q' &= -c^{-1}(1-\gamma Q + O(\varepsilon))\Omega((1-\gamma^{-1})P + Q + O(\varepsilon)) \\ P' &= c(-\gamma^{-1}P + Q + O(\varepsilon)) \end{aligned}$$

- **M_ε depends on ε smoothly**
- **M_ε is attracting and contains $(\gamma^{-1}, 1, 1, 0)$, $(0, 0, 0, 1)$ $\Longrightarrow$**

  For sufficiently small $\varepsilon > 0$, any heteroclinic connecting $(\gamma^{-1}, 1, 1, 0)$ to $(0, 0, 0, 1)$ must lie in M_ε

Orbit construction on M_0 and M_ε

- on M_0 add a direction corresponding to the velocity c

$$
\begin{aligned}
Q' &= c^{-1}(\gamma Q - 1)\Omega(Q + (1 - \gamma^{-1})P) \\
P' &= -c\gamma^{-1}P + cQ \\
c' &= 0
\end{aligned}
$$

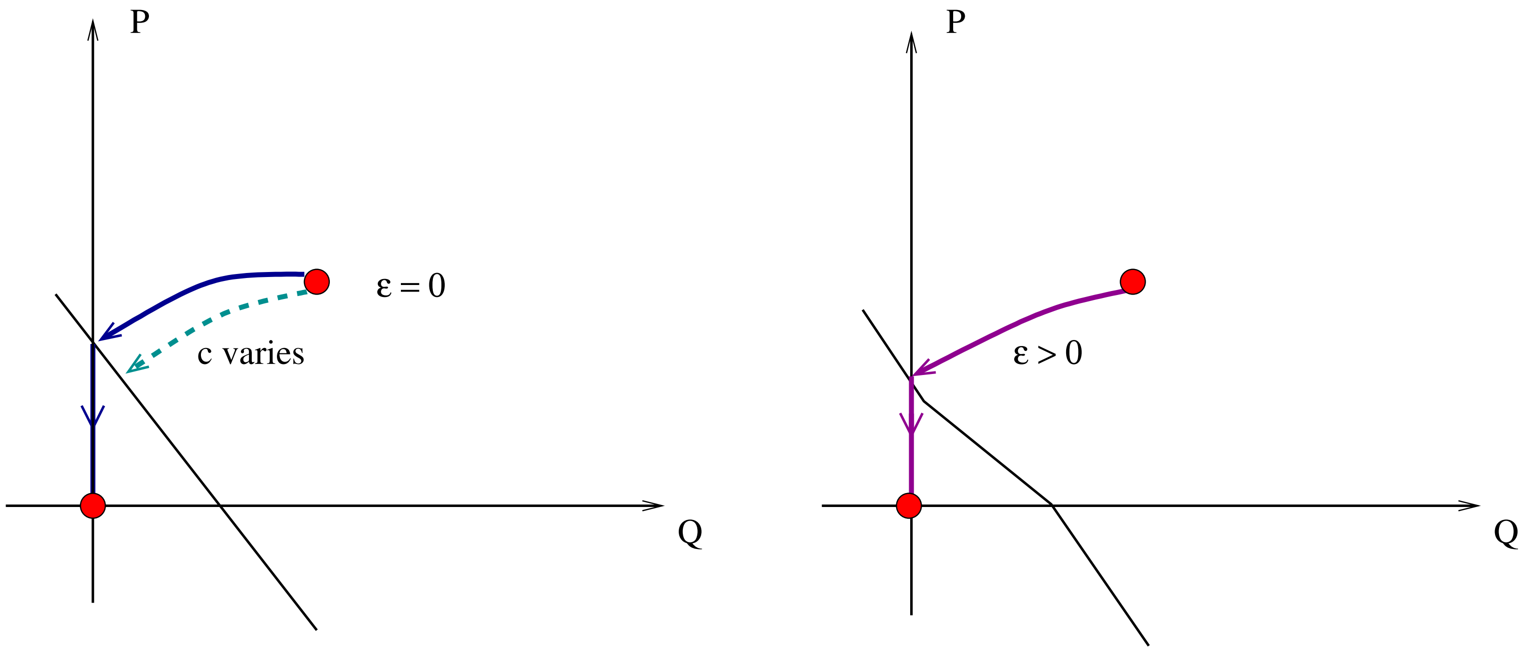
- the front is constructed as a transversal intersection of two suspended in $M_0 \times \{c \sim c_0\}$ invariant manifolds
- upon switching on a sufficiently small $\varepsilon > 0$ the transversal intersection perturbs with a nearby c_ε replacing c_0

$\Longrightarrow$

a unique up to translation front for the perturbed system exists

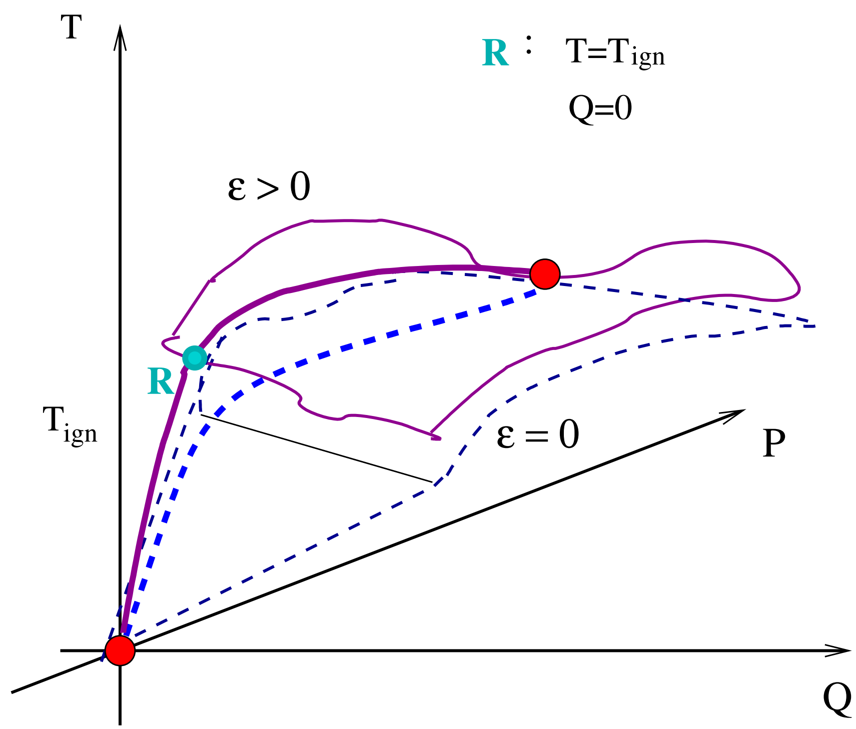
Transversal intersection

1-dim W^u of $(Q, P) = (\gamma^{-1}, 1)$ intersects transversely 1-dim W^s of $(Q, P) = (0, 0)$ as the speed parameter c varies



The argument is independent of the smoothness of $\Omega(T)$

Discontinuous $\Omega(T)$



- the manifold has precisely one point R with $T = T_{ign}$, $Q = 0$
- the front is continuous, monotone, and goes through R
- the transversality argument: there is a unique c s.t. W^u reaches R
- the system is linear for temperatures below ignition

Conclusions

Model for Combustion in Porous Media

Methods of geometric singular perturbation theory $\Longrightarrow$

- **for** $\varepsilon > 0$ **but small, front exists**
- **for** $\varepsilon > 0$ **but small, front is unique**
- **fronts converge as** $\varepsilon \to 0$ **to the front of unperturbed system**

Model of high Lewis number combustion (high density fluids burning at high temperatures)

$$
\begin{aligned}
u_t &= u_{xx} + y e^{-1/u} \\
y_t &= \frac{1}{\mathrm{Le}} y_{xx} - \beta y \Omega(u)
\end{aligned}
$$

$\beta > 0$**- exothermicity,** $\Omega(u) = e^{-1/u}$ **for** $u \geq 0$ **and** $\Omega(u) = 0$ **otherwise.**

The same approach works and similar results are obtained [G., Jones]

Stability of Fronts

High Lewis number combustion:

Front of the model with $\mathrm{Le} = \infty$ is STABLE/UNSTABLE $\Longrightarrow$ Front of the model with $[\mathrm{Le} >> 1]$ is STABLE/UNSTABLE

- **Construction of the Unstable Bundle**

Stability of the Front in CPM - work in progress