(II) Natasha Sesum. (joint with P. Daskalopoulos)

$$\frac{du}{dt} = \Delta \log u \quad \text{on } R^2 \qquad\qquad \frac{du}{dt} = \Delta(u^m) \quad \text{on } R^N$$

($m<1$, fast diff) $\quad N \geq 3$

$m = \frac{N-2}{N+2}$

- Geometric representation.

(i) $N=2$. $\quad R_{ij} = \frac{1}{2} R g_{ij}$

$g = u\, ds^2$.

$$R = -\frac{\Delta \log u}{u}$$

$$\frac{d g_{ij}}{dt} = -2R_{ij} \quad \text{on } R^2.$$

$$\Updownarrow$$

$$\frac{du}{dt} = \Delta \log u$$

(ii) $N \geq 3$. $\quad g = u^{\frac{N}{N+2}} ds^2$.

$$R = -\frac{\Delta(u^m)}{u} \qquad m = \frac{N-2}{N+2}.$$

$$\frac{dg_{ij}}{dt} = -R g_{ij} \iff \frac{du}{dt} = \Delta(u^m)$$

Goal: to study the asym. bhv. of u near the vanishing time $\iff$

to study the singularity of the geo. flow at $T < T_0$.

Log. fast diff:

$$\int_{R^2} u_0 < +\infty \Rightarrow u_0(x) \to 0 \quad \text{as } |x| \to \infty.$$

Nonuniqueness.

Thm. (Das, del, Pino): For every u_0, and $\forall \lambda \geq 2$, there is a solution to $\frac{du}{dt} = \Delta \log u$, $u(x,0) = u_0$ s.t. $T_\lambda = \frac{1}{2\pi\lambda} \int u_0$

$\lambda = 2 \iff$ maximum solutions

(Esteban, Rodrigues, Vasquez) If supp $u_0 \subset B(0,a)$, then

$$u(x,t) = \frac{2t}{|x|^2 \log^2 |x|}(1 + o(1)) \qquad |x| \gg 1, \quad t \in [0,T)$$

$\to \log$.

(J. R. King). formal Asym. of the vanishing time of u.

Inner region: $\log r (T-t) < T$ our solution u vanishes exp.

and after approximating rescaling converge to a cigar

$$\left(u = \frac{1}{\frac{T}{2}|x|^2+1} \right)$$

Outer region: $\log r (T-t) > T$ log. cusp. persists.

Inner region:

$$\hat{u}(x,\tau) = \tau^2 (x,t) \qquad \tau = \frac{1}{T-t}.$$

$$\frac{d}{d\tau} \hat{u} = \Delta \log \hat{u} + \frac{2L'(\tau)}{L(\tau)} \operatorname{div}(y \cdot \hat{u}) + \frac{2\hat{u}}{\tau}$$

Thm (Das): If supp $u \subset B(0,a)$, then $\hat{u}(x,t) \longrightarrow \frac{1}{\frac{T}{2}|x|^2+1}$.

Outer region:

$$v(z,\theta,t) = \tau^2 u(r,\theta,\tau). \qquad z = \log r.$$

$$\hat{V}(z,\theta,\tau) = \tau^2 V(z,\tau,\theta,t). \quad \tau = \frac{1}{T-t}$$

$$\tau \hat{V}(z,\theta,\tau) = \frac{1}{2}(\log \hat{V}) + \hat{V}_z + 2\hat{V}.$$

Thm: If supp $u \in B(0,a)$, then $\hat{V}(g(t))$ converges to a steady stable of the above equa.

$$V(s) = \begin{cases} {}^{2\tau}/_{s^2} & z \geq T \\ 0. & z < T \end{cases}$$

Necessary ingredient:

Geometric width, $g = u ds^2$

$w(g)$. $F: R^2 \to [0,\infty)$. proper fun.

$$\inf_F \sup_c \{F = c\} := w(g)$$

$$w(g) = \sup_{r>0} 2\pi r \sqrt{u}$$

Thm. (Das Ham.)

$$c(T-t) \le w(q) \le C(T-t)$$

$$\frac{c}{(T-t)^2} \le \max R \le \frac{C}{(T-t)^2}.$$

$\Rightarrow$ Type II singularity that RF develops in finite time.

- Yau-Hamilton Harnack Est. for R:

$\forall E, L,\ \exists A, B, C, D > 0$ s.t. if $t=0$, $R \ge I - E$ (in dim 2)

$$\frac{\partial R}{\partial t} - \frac{|DR|^2}{(R+E)^2} \ge -L \quad \text{then}$$

$$\frac{\partial R}{\partial t} - \frac{|DR|^2}{(R+E)^2} + F\left(R+E, \frac{|DR|^2}{(R+E)^2}\right) \ge 0, \quad \forall t > 0$$

$\forall x_1, x_2 \in \mathbb{R}^2,\ t_1 < t_2$.

$$\frac{1}{\sqrt{R+E}}(x_1, t_1) \ge \frac{1}{\sqrt{R+E}}(x_2, t_2) - C_1(t_2 - t_1) - C_2 \cdot \frac{\mathrm{dist}^2_{t_1}(x_1, x_2)}{t_2 - t_1}$$

- Aronson-Benilan Inequa.

$$u_t \le \frac{u}{t} \iff R \ge -\frac{1}{t}$$

- Almost monotone property (Cafferella, Arson)

If $\operatorname{supp} u_0 \subset B(0, a)$, then $u(x,t) \ge u(y,t)$ $\forall t \in [0,T)$

$\forall x, y$ $|y| \ge |x| + a$ (by reflection principle).

Pf: $\bar{u}(x,t) = \tau^2 u(x,t)$, $\tau = \frac{1}{T-t}$.

$\tau_k \to +\infty$,

$\hat{u}_k(x,\tau) = \alpha_k\, \bar{u}(\alpha_k x, \tau + \tau_k)$, $\alpha_k = [\bar{u}(0, \tau_k)]^{-1}$.

$\hat{u}_k(0,0) = 1$, $\tau \in [-\tau_k + \frac{1}{T}, +\infty)$

$$\frac{d}{d\tau}\bar{u}_k = \Delta \log \bar{u}_k + \frac{\bar{u}_k}{\tau + \tau_k}$$

$$\bar{R}_{max} = R_{max}\,(T-t)^2 \leq \frac{c}{(T-t)^2}\,(T-t)^2 = c.$$

$$(R_{max})_k \geq -\frac{c}{(\tau+\tau_k)^2}$$

$$\frac{\partial}{\partial t}\bar{u}_k = \Delta \log \bar{u}_k + \frac{\bar{u}_k}{\tau+\tau_k}$$

$$\bar{u}_k \longrightarrow u(x,\tau)$$

$$\frac{d}{dt}u = \Delta \log u. \qquad \forall \tau \in (-\infty, +\infty).$$

R width is bounded. at each time slice.

Thm (Das. Sesum): If $g = u\,ds^2$ is complete. eternal solution to the R.F. Eq. with bounded R and width of each time slice $\Rightarrow$ $u(x,t) = \dfrac{2}{\beta(x-x_0)^2 + C\,e^{2\beta t}}$

Pf. $v(s,\theta,t) = r^2\,u(r,\theta,t)$ $\quad (s = \log r)$.

$$v_t = \Delta_c \log v.$$

$$V(s,t) = \int_0^{2\pi} \log v\,d\theta$$

$$R > 0 \Rightarrow V_{\infty} \leq 0$$

key esti. $\quad c(t) \leq v(s,\theta,t) \leq C(t)$

- Harnack esti. for R.

$$\left(\frac{\partial R}{\partial t} - \frac{|\nabla R|^2}{Ru} \geq 0\right) \Rightarrow R \to 0. \quad \text{as } |x| \to \infty$$

$$X = \nabla R + R\,\nabla f. \quad f = -\log u$$

$$M_{ij} = \nabla_i\nabla_j f - \frac{1}{2}\Delta f\, g_{ij}$$

$$\int_{B_\rho}\left(\frac{|X|^2}{R} + \frac{|M_{ij}|^2}{u}\right)dx \leq I_\rho + J_\rho \to 0 \quad \text{as } \rho \to +\infty$$

integrals over ∂B_ρ

$M_{ij} = 0.\ X = 0 \Rightarrow g$ is a gradient soliton on R^2.

L.F. Wu. (g is either a cigar or a standard plane)

$\max(g) \le C \Rightarrow$ Cigar.

$u(x, t_k) \simeq (T-t_k)^2 / \lambda|x|^2 + \lambda_k$ $\quad |x| < \lambda_k^{1/2} \cdot M$

$\lambda = \frac{T}{2}$. (by Hamilton Harnack esti.).

Fast diff. equa on R^N. ($N \ge 3$).

If $\frac{\partial u}{\partial t} = \Delta(u^m)$, $\quad m = N-2/N+2$

$$t \in [0, T) \quad \begin{matrix} u(x,0) = u_0 \\ T < +\infty \end{matrix} \quad \Leftrightarrow \quad g = u^{\frac{N}{N+2}} ds^2$$

Sufferent similar solution to $(*)$. $\quad \frac{\partial}{\partial t} g_t = -R g_t$.

and it satisfies $\quad \Delta f^m + |r| \operatorname{div}(x \cdot f) = 0$

$$u(x,t) = (T-t)^n f\left(\frac{|x|}{(T-t)^n}\right)$$

Barenblatt Solutions.

$$B_k(x,t) = \left(\frac{(T-t)\, e^t}{|x|^2 + k(T-t)^{2r}}\right)^{\frac{1}{1-m}}$$

Comparison Thm. $\quad u_1 \le u_2$ at $t=0$ $\Rightarrow u_1(x,t) \le u_2(x,t)$

If u_0 has a sufferent decay. $\Rightarrow$ we fall in the category of compact Yamabe flow.

Del. Rieno. Saez. $\quad u_0 \le \frac{C}{|x|^{N+2}} \quad |x| \gg 1$.

$$\Rightarrow u(x,t)(T-t)^{-\frac{1}{1-m}} \longrightarrow \left(\frac{C}{\lambda^2 + |x - x_0|^2}\right)^{N+2/2}$$

$B_{k_2} \le u_0 \le B_{k_1} \Rightarrow u(x,t)$ vanishes at T

have the same vanishing time.

$$\hat{u}(x,\tau) = (T-t)^{-\beta}\, u(x_0\,(T-t)^{\eta},\, t), \qquad \tau = -\ln(T-t)$$

$$\frac{d}{\partial\tau}\,\hat{u} = \Delta \hat{u}^{m} + |r| \cdot \mathrm{div}(x \cdot \vec{v}).$$

Thm (Das. Sesum): If $2 \le N \le 5$, then $\hat{u}(x,\tau)$ converges to a Barenblatt B_{K_0} ($\int_{R^2} (u_0 - B_{K_0})\, dx = 0$).

Thm. ... If $N \ge 6$, if $u_0 = B_{K_0} + k_1 f \in L^1(R^N)$ then $\hat{u}(x,\tau)$ as $\tau \to \infty$ converges to $\hat{B}_{K_0}$.

Pf. relied on L^1-comparison principle.

$$B_{K_1} - B_{K_2} \in L^1(R^N), \quad \text{iff} \quad 2 \le N \le 5$$

- Comparison principle.
  - If $u_0 - v_0 \in L^1(R^N)$, $\Rightarrow u(x,t) - v(\cdot,t) \in L^1(R^N)$
  - If $t > s$, $\Rightarrow \int_{R^N} |u(x,s) - u(x,s)|\, ds \le \int_{R^N} |u(x,t) - v(x,t)|\, dt$
  - Recently, remove the assumption $u_0 \le B_{K_1}$ the bound from above is crucial

Thm (Das, ...): $\exists$ a class of rapidly symmetric solutions $u(x,t)$ with $u(x_0,t_0) = \frac{c^* T}{|x|^{N+2/N}}(1 + o(1))$, $|x| \gg 1$.

then

- the vanish time of $u(x,t)$, $T^* > T$.
- $u(x,t) = \frac{c(t)}{|x|^{N+2/N}}(1 + o(1))$. $t < T$
- For $t \in (T, T^*)$, $u(x,t) \le \frac{c(t)}{|x|^{N+2}} \Rightarrow$ the vanishing profile is given by a sphere

(iii):

$$\int_{R^N} |u(x,t) - B(x,t)|\, dx \le \int_{R^N} |u(x,0) - B(x,0)| \le c.$$

$t = T$

$$\Downarrow$$

$$u(x,t) \in L^1(R^N) \underset{\text{lemma}}{\Longrightarrow} u(x,t) \le \frac{c(t)}{|x|^{N+2}}.$$

Pf (of the lemma):

$$\frac{\partial}{\partial t} v = \Delta(v^m)$$

$$v(t) = v(0)/t = \Delta \int_0^t v^m.$$

$$\Delta w \ge f$$

$$Z = \int_r^{+\infty} \frac{1}{s^{N-1}} \int_{|x| \le \rho} f$$

$$\Delta w \ge \Delta Z.$$

$$\Delta(w - Z) \ge 0.$$

$$w - Z \le \oint (w - Z) \le \oint w$$

$$w \le Z \le c/r^{N-2}.$$

Recent: If g is conformally flat, Yamabe flow in R^N.

($u \le B$) s.t. $Rm > 0$, R is bound at each time slice

then the singularity is always of type (I).