Monday (3-12-07) : 2:00pm – 3:00pm by Felix Schulze.

Aim: to give a flow proof of Euclidean isoperimetric inequa.

$U \subset \mathbb{R}^{n+1}$. U: open and bd, ∂U smooth, connected.

Isoperimetric inequa.

$$|\partial U|^{n+1/n} \geq C_{n \times n} |U|.$$

can replace U by its outer minimizing hull Ω

i.e. a minimizer of $\inf \{ |\partial V| \mid U \subset V \subset \mathbb{R}^{n+1} \}$

$$\Rightarrow |\partial \Omega| \leq |\partial U|, \quad |\Omega| \geq |U|.$$

Can show this outward minimizing hull exists.

and for $n+1 \leq 7$, $\partial \Omega$ is $C^{1,1}$, with $H_{\partial\Omega} \geq 0$

$\Rightarrow$ suffices to prove the isoperimetric inequa. for Ω

By approximating we can assume $\partial\Omega \in C^\infty$, $H_{\partial\Omega} > 0$

<u>H^k-flow</u>, $T: M \times [0, T) \to \mathbb{R}^{n+1}$.

$$\begin{cases} \frac{\partial T}{\partial t} = -H^k \nu, \\ T(\cdot, 0) = T_0 \quad \text{s.t.} \quad T_0(M) = \partial\Omega. \end{cases}$$

- $k=1$ M.C.T.
- $H_{\partial\Omega} > 0 \Rightarrow$ short time existence.
- $H > 0$ preserved $\Rightarrow$ unknown.

$A(t)$: the Area, $V(t)$: the enclosed volume.

$$\frac{\partial}{\partial t} V = -\int_{M_t} H^k d\mu, \quad \frac{\partial}{\partial t} A = -\int_{M_t} H^{k+1} d\mu$$

Compute:

$$\frac{d}{dt}\left[A^{\frac{n+1}{n}} - C_{n+1} V \right] \leq \cdots \leq 0 \qquad (k \geq n-1)$$

use Hölder Inequa.

Geometric estimate:

$N^n \subset R^{n+1}$, hypersurface, C^2, closed.

$N^+ = \{ pt.\ p \in N \}$ N can be reduced by a plane from outside.

Note. $\nu(N^+) = S^n$ and $\lambda_1(p), \ldots, \lambda_n(p) \geq 0$ $\forall p \in N^+$

$$|S^n| = \int_{S^n} 1 \, dx \leq \int_{N^+} \nu^+ \, dx = \int_{N^+} |\det D\nu| \, d\mu$$

$$\leq \frac{1}{n^n} \int_{N^+} H^n \, d\mu$$

$$\leq \frac{1}{n^n} \int_M |H|^n \, d\mu$$

If the flow exists smoothly with $V(T) = 0$

$$\Rightarrow A^{\frac{n+1}{n}}(0) = C_{n+1} V(0) \geq A^{\frac{n+1}{n}}(T) + C_{n+1} V(T) \geq 0$$

Calculation also shows if we replace R^3 by (N^3, g)

where M is (hypersurface) simply connected, complete, non-positive

sectional curvature.

By Bonnet formula.

$$|S^2| - 4S = \int_{M^2} G + K_{min} \, d\mu \leq \frac{1}{4} \int_{M^2} H^2 \, d\mu$$

(B' Kleme '91): (N^3, g) as above

$\Rightarrow$ the Euclidean isoperimetric inequa holds for (N^3, g).

Problem: Singularities.

Idea: level-sets formula. i.e. $u : \Omega \to R^+$, $u = 0$ on $\partial\Omega$.

$M_t = \{ u = t \}$

$$(*): \quad \operatorname{div}\left(\frac{Du}{|Du|} \right) = \frac{-1}{|Du|^{1/n}} \quad \text{on } \Omega.$$

$(*_w)$: $\quad div\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\right) = -\frac{1}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\,k$

elliptic regularities,

Interpreta: $\quad N_t^\varepsilon = graph\left(\frac{u^\varepsilon}{\varepsilon} - \frac{t}{\varepsilon}\right) \subset \Omega\times R$, level sets

$u^\varepsilon(x,z) = u^\varepsilon(x) - \varepsilon z \longrightarrow u(x,z) = u(x)$

• $(*_w)$. $\quad \Rightarrow \quad u^\varepsilon$ satisfies $(*)$

$\Rightarrow \quad N_t^\varepsilon$ are solutions (smooth translating) to H^n flow

Hope $N_t^\varepsilon \xrightarrow{\varepsilon\to 0} \{u=t\}\times R$

Show: existence to solutions of $(*_w)$, $\varepsilon>0$, uniformly C^1-bds

$\Rightarrow$ Define a weak solution to $(*)$ by taking $\varepsilon_i \to 0$

then $u = \lim_{\varepsilon_i\to 0} u^{\varepsilon_i}$ in L^0, $u\in C^{0,1}(\Omega)$, $u=0$ on $\partial\Omega$.

Thm. $\Omega \subset N^{n+1}$, open bded, $H_{\partial\Omega}>0$ if $n=2$

let $k\geq 1$ and (N,g) complete, simply connected 3-manifold

non-positive sectional curvature. If $n\geq 4$, let $N^{n+1} = R^{n+1}$.

and $k > n-1$. If h is a weak sol to $(*)$

then $\quad \mathcal{H}^n(\partial^*\{h>t\})^{\frac{n+1}{n}} - C_{n+1}\mathcal{H}^{n+1}(\{h>t\})$

is a monotone, increasing for $[0, \sup u]$

Corollary $\quad n+1 \leq 8 \quad \Rightarrow$ Isop. inequa

And prove Klein's thm

Properties.

- $\{u > t\} \subset \Omega$, minimize area from the outside in Ω.
- for a.e. t $N_t = \{u = t\} \times R$.
  is a unit density n-rectifiable manifold and.

  $$N_t^{\varepsilon} \longrightarrow N_t \quad \text{as Radon measures}$$

  And $\exists$ subsequence $\varepsilon^{(i)}(t)$ s.t.

  $$N_t^{\varepsilon_j(t)} \longrightarrow N_t \quad \text{as manifolds}$$

  $$\frac{\partial}{\partial t} A = -\int_H {}^{k+1} \Rightarrow$$

  $$\int_0^T \int_{N^s \wedge (\Omega \times \mathbb{Z})} H^{n+1} \, d\mu_t \, dt \le C.$$

  Fatou's lemma: $\Rightarrow \int_0^T \int_{N_t \wedge (\Omega \times \mathbb{Z})} H^{k+n} \, d\mu_t \, dt < C.$

  and $|H|^{\varepsilon_i} d\mu_{\varepsilon_i} \longrightarrow H \nu \, d\mu$

- For $k > n-1$.
  $\Gamma_t = \{u = t\}$ is $C^{1,\alpha}$ up to a sup set $S \subset \Gamma_t$
  with $\mathcal{H}^{\lambda}(S) = 0$. for a.e. t.

  $$\int_{\Gamma} H^{k+1} \, d\mu_t < \infty$$

- Volume-fattening: $v(t) = |\{u = t\}|$ is a Hölder-cont in time.

  $$\frac{\partial}{\partial t} \int v = -\int H^k.$$

- Can pass the monotonicity calculation to a lim[t] by
  a lower-summation argument.

for a.e. t $\displaystyle\int_{\Gamma_t = \{u = t\}} H^n \geq 4^n |S^n|$.

use $\displaystyle\int_M \mathrm{div}_M(X) = -\int_M \langle X, H \rangle$

for 2-dim. surfaces

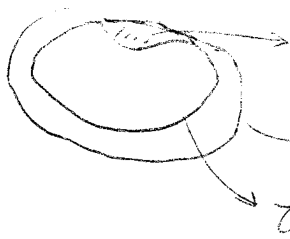
$\underline{n \geq 3}$: $N^{n+1} = R^{n+1}$

for a.e. t

$\Gamma_t = \{u = t\}$ are $C^{1,\alpha}$ up to a closed sing set

S with $\mathcal{H}^*(S) = 0$ and $\int_{\Gamma_t} |H|^{n+1} \, d\mu < \infty$

Idea:



C = convex hull of Γ_t.

$C_s = \{x \in R^{n+1} \mid \mathrm{dist}(x, C) = s\}$

Γ_t $\quad C_s$ are $C^{1,1}$.

$\pi_s : C_s \to C$: the closest point projection.

Claim: $H_{C_s}(p) \leq H_{\Gamma_t}(\pi_s(p))$ a.e. $\pi_p \in \Gamma_t^+$.

Prob:

C_s, C $>$ integral coincides.

can show (+) $\displaystyle\int_{C_s \cap \pi_s^{-1}(A \cap \Gamma_t)} f(\pi_s(p)) \, d\mathcal{H}^n(p) \leq \int_{\Gamma_*} f(q) \left(1 + \frac{1}{n+1} |H(q)| s\right)^{n+1} d\mathcal{H}^n(q)$