3-13-07. (Tuesday) By Klaus Ecker.

Perelman (Σ, g)

$$W(g,f,\tau) = \int_\Sigma \left(\tau(|\nabla f|^2 + R) + f - n \right) u \, dV$$

Also for Non-compact

where $u = \dfrac{e^{-f}}{(4\pi\tau)^{n/2}}$

$\inf \to 0$.

$f: \Sigma \to R$

$\tau > 0$

Entropy: $\mu(g,\tau) = \inf \{ W(g,f,\tau) \}$ s.t. $\int_\Sigma u \, dv = 1$

Idea: Suppose $\frac{\partial g}{\partial t} = -2Ric$ (Ricci flow)

$\frac{\partial \tau}{\partial t} = -1$.

Then $\frac{\partial}{\partial t} \mu(g(t), \tau(t)) \geq 0$

$\longrightarrow \cdots \checkmark \xrightarrow{\text{properties of } \mu}$

Thm. Suppose $(\Sigma, g(t))$ satisfy RF for $[0,T)$, $T < \infty$

then $\exists\, \kappa = \kappa(n, g(0), T) > 0$ s.t.

$$\mathrm{Vol}(B_r^t(x_0)) / r^n \geq \kappa > 0$$

$\forall\, t \in [0,T)$, $\forall\, r \leq \sqrt{T}$, whenever $r^2 |Rm| \leq 1$ in $B_r^t(x_0)$.

$\Omega \subset R^{n+1}$ open, $f: \Omega \to \Omega$, $\beta: \partial\Omega \to R$, $\tau > 0$.

$$W_\beta(\Omega, f, \tau) = \int_\Omega (\tau|\nabla f|^2 + f - (n+1))\, u \, dx + 2\tau \int_{\partial\Omega} \beta u \, ds$$

$$u = \frac{e^{-f}}{(4\pi\tau)^{n+1/2}}$$

$$\mu_\beta(\Omega,\tau) := \inf \{ W_\beta(\Omega,f,\tau),\ \int_\Omega u\,dx = 1 \}$$

Properties of $\mu_\beta(\Omega,\tau)$:

(1) $\mu_\beta(\Omega,\tau) \geq -C(n, C_S(\Omega), \sup_{\partial\Omega}|\beta|)\,(1 + \log(1+\tau))$

(Log. Sobolev inequality for $\Omega \subset R^{n+1}$).

② Suppose that $Ric_\beta(n, \tilde{r}) \geq -C_0 > -\infty$. Then if

$$\frac{|\Omega \cap B_r(x_0)| + r^2 \int_{\partial\Omega \cap B_r(x_0)} |\beta|\, ds}{|\Omega \cap B_{\frac{r}{2}}(x_0)|} \leq C_1 \quad \text{and} \quad |\Omega \cap B_{\frac{r}{2}}(x_0)| > 0,$$

then

$$\frac{|\Omega \cap B_r(x_0)|}{r^{n+1}} \geq K > 0, \quad \text{where } K(n, C_0, C_1) = K_0$$

"Bad sets": <u>Local volume ratio collapse.</u>

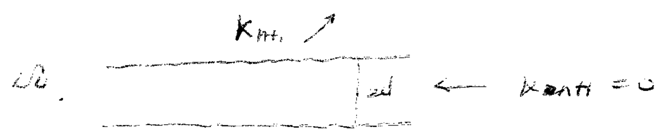
<u>Define</u>: $\exists$ a sequence $\{B_{r_k}(x_k)\}$ w/ $|\Omega \cap B_{\frac{r_k}{2}}(x_k)| > 0$.

$$\& \quad \frac{|\Omega \cap B_{r_k}(x_k)| + r^2 \int_{\partial\Omega \cap B_{r_k}(x_k)} |\beta|\, ds}{|\Omega \cap B_{\frac{r_k}{2}}(x_k)|} \leq C_1$$

but

$$\lim_{k \to \infty} \frac{|\Omega \cap B_{r_k}(x_k)|}{r_k^{n+1}} = 0$$

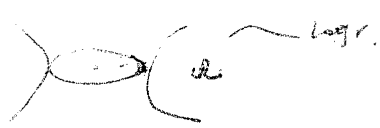
①. Slab



$$\frac{|\Omega \cap B_r|}{|\Omega \cap B_{r/2}|} \leq C(n, d) \quad \forall r > 0 \qquad \beta = H_{\partial\Omega} = 0$$

$$\lim_{r \to \infty} \frac{|\Omega \cap B_r|}{r^{n+1}} = 0.$$

②. $\Omega \subset \mathbb{R}^3$. $|x_3| < \cosh^{-1}\sqrt{x_1^2 + x_2^2}$.



catenoid.

$$\beta = H_{\partial\Omega} = 0$$

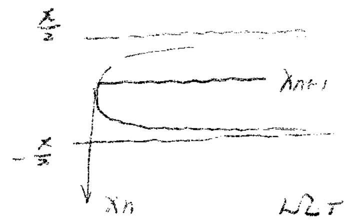
$$\frac{|\Omega \cap B_r|}{|\Omega \cap B_{r/2}|} \leq C_1 \quad \forall r > 0$$

$$\frac{|\Omega \cap B_r|}{r^3} \leq \frac{Cr^2(1+\log r)}{r^3} \longrightarrow 0 \quad \text{as } r \to \infty.$$

"smoother"

$G \times R^{n-1} = \Omega$. G: inside of the grim reaper curve.

G:



$x_{n+1} > -\log \cos x_n$

$\Omega_T = \Omega + t e_{n+1}$

G moves by mean curvature under transformation.

$\beta(x) = H_{\partial\Omega}(x) = e^{-x_{n+1}}$.

$\exists\ B_{r_k}(x_k)$ with $\beta(B_{r_k}(x_k)) \le 1$.

and $\frac{|\Omega \cap B_{r_k}(x_k)|}{r_k^{n+1}} \to 0$

$\boxed{\text{In } \Omega = G, \quad \inf_{\tau>0} \mu_\beta(\Omega,\tau) = -\infty}$

<u>Lemma</u>: The minimizer f for $\mu_\beta(\Omega,\tau)$ satisfies

$w(f) = \tau(2\Delta f - |\nabla f|^2) + f - (n+1) \equiv$ const. in Ω.

and $\nabla f \cdot \nu = \beta$ on $\partial\Omega$.

If a funct. f satisfies $\nabla f \cdot \nu = \beta$ on $\partial\Omega$.

then $W_\beta(\Omega, f, \tau) = \int_\Omega w u \, dx$

$\nabla f \cdot \nu = \beta \Rightarrow \nabla u \cdot \nu = -\beta u \Rightarrow 2\tau \int_{\partial\Omega} \beta u \, ds = -2\tau \int_{\partial\Omega} \nabla u \cdot \nu \, ds$

$= -2\tau \int_\Omega \Delta u \, dx$

$\Delta u = u\left(|\nabla f|^2 - \Delta f\right)$.

Assume now that (Ω_t) satisfies

$-\frac{\partial X}{\partial t} \cdot \nu = \beta$ on $M_t = \partial\Omega_t$ ($\beta = H \to$ MCF)

and let f satisfies $\frac{\partial f}{\partial t} + \Delta f = |\nabla f|^2 + \frac{n+1}{2\tau}$ in Ω_t

with $\nabla f \cdot \nu = \beta$ on $M_t = \partial\Omega_t$.

and change by diffeomorphism: $\frac{\partial X}{\partial t} = -\nabla f(x,t)$ in Ω_t *

<u>Note</u>: $\longrightarrow \frac{\partial X}{\partial t} = -\beta\nu - \nabla^{M_t} f$ on M_t

Proposition 1: Ω_T, $f(t)$ as above, $\frac{\partial \tau}{\partial t} = -1$

$$\frac{\partial}{\partial t} W_\beta(\Omega_T, f(t), \tau(t)) = 2\tau \int_{\Omega_T} \left| \nabla_i \nabla_j f - \frac{\delta_{ij}}{2\tau} \right|^2 u\, dx - \int_{\partial\Omega_T} \nabla w \cdot \nu\, u\, ds$$

proof involves Perelman's identity.

$$\left(\frac{d}{dt} + \Delta\right) w = 2\tau \left| \cdots \right|^2 + \nabla w \cdot \nabla f.$$

$$\left(\text{Note: } \frac{\partial}{\partial t} \int_{\Omega_T} u\, dx = 0 \right)$$

Prop 2: $-\nabla w \cdot \nu = 2\tau \left(\frac{\partial \beta}{\partial t} - 2 \nabla^M \beta \cdot \nabla^M f + A(\nabla^M f, \nabla^M f) - \frac{\beta}{2\tau} \right)$

2^{nd} f. f. of M_T

Harnack type expression

Hamilton: $Z(V) = \frac{\partial H}{\partial t} - 2 \nabla^M H \cdot V + A(V, V)$. V: tangent to M_t.

for $\beta = H$. Harnack inequ.

$$Z(V) + \frac{H}{2\tau} \geq 0. \quad \text{with "=" on expanding solition of M.C.F.}$$

$Z(V) = 0$ on translating solution for a suitable V.

$$\left(\text{for tangent solution. } u = e^{x_{n+1} - \tau} \right)$$

Observation: $\tau = a - t$.

$Z(\nabla^M f) - \frac{H}{2\tau} = 0$ on shrinking solitons of MCF.

with $f = |x|^2/4\tau$

$$\beta = H = \frac{x \cdot \nu}{2\tau}$$

Conjecture: $$2\tau \int_{M_T} \left(Z(\nabla^M f) - H/2\tau \right) u\, ds \geq 0$$

on compact with $H \geq 0$. Better: $Z(\nabla^M f) - \frac{H}{2\tau} \geq 0$. $\forall V$: tang on compact $(H \geq 0)$. (M_t).

<u>Conclusion</u>. Should the conjecture be true, then

$\beta = H$.

$$\frac{\partial}{\partial t} W_H(\Omega_t, f(t), \tau(t)) \geq 0.$$

$\rightsquigarrow \cdots \longrightarrow u_H(\Omega_{t_1}, a-t_1) \leq u_H(\Omega_{t_2}, a-t_2)$.

$\forall\ 0 \leq t_1 \leq t_2 < T$.

$\rightsquigarrow\ u_H(\Omega_T, r^2) \geq u_H(\Omega_0, t+r^2) \geq -c_0(n, \Omega_0, \sup_M |H|, T)$

if $r \leq \sqrt{T}$ $\quad t \leq T_0$

property ↑ for Ω_0

$\rightsquigarrow$ if conjecture holds, then $\exists\ K = K(n, T, \Omega_0, \sup_{M_0} H)$

property:

s.t. $\dfrac{|\Omega_t \cap B_r(x_0)|}{r^{n+1}} \geq K$ $\qquad \forall\ t \in [0, T]$, $\forall\ r \leq \sqrt{T}$

where $|\Omega_r \cap B_{r/2}(x_0)| > 0$.

and $\dfrac{|\Omega_t \cap B_r(x_0)| + r^2 \int_{\partial\Omega \cap B_r(x_0)} |H|^2\, ds}{|\Omega_\tau \cap B_{\frac{r}{2}}(x_0)|} \leq C_1$

let (M_s') be a blow-up limit of MCF about (x_0, t_0).

$(M_s') \longleftarrow \lim_{\lambda \downarrow 0} \frac{1}{\lambda}(M_{\lambda t + t_0} - x_0)$

then $\dfrac{|M_s' \cap B_r(0)|}{r^{n+1}} \geq K > 0$

some k as above.

but $\forall s < \infty$, $\forall r > 0$.

so slab caten given (?) reeper. can't be blow-up limits!

$$2\tau \int_{\partial\Omega} \beta u\, ds = -2\tau \int_{\partial\Omega} \partial u\, v\, ds = -2\tau \int_\Omega \Delta u\, dx$$

$$= 2\tau \int_\Omega (\Delta f - |\nabla f|^2)\, u\, dx$$

$$= \int_\Omega \underbrace{(\tau(2\Delta f - |\nabla f|^2) + f - (n+1))}_{w} u\, dx$$