3-14-07: by Dan Knopf:

Ecker, K. N. Topping (Crelle):

$(M^n, g(t))$, $\alpha \le t \le \omega$, $\frac{\partial}{\partial t} g = 2h$

$D = \frac{\partial}{\partial t} - \Delta$

$D^* = \left(\frac{\partial}{\partial t} + \Delta + \operatorname{tr}_g h\right)$.

$\frac{\partial}{\partial t} \int \varphi \psi \, d\mu = \int [(D\varphi)\psi - \varphi(D^*\psi)] \, d\mu$

Eg #1. (R^n, g), $h \equiv 0$

$\psi(x,t) = [4\pi(\tau - t)]^{-\frac{n}{2}} e^{-\frac{|y-x|^2}{4(s-t)}}$ $\quad (t < s)$

$D\varphi = 0$

$\int_{R^n} \varphi(x,t)\, \psi(x,t) = \varphi(y,s)$

Eg #2. $F_t : M^n \hookrightarrow M_t^n \subset R^{n+1}$.

$\frac{\partial F}{\partial t} = -HV$.

$h = -HA$.

$D^*\psi = \left| \frac{(x-y)^\perp}{2(s-t)} - HV \right|^2 \psi$

$\int_{M_t^n} \psi \, d\mu \quad \searrow$ in t.

$\Theta_{MCF} = \lim_{t \nearrow 0} \int_{M^n_t} \psi \, d\mu$

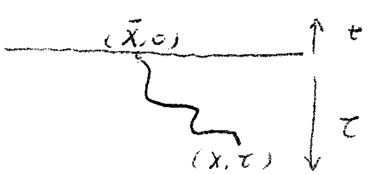
Eg #3. M^n: compact. $h = -Ric$

$\psi = [\tau(2\Delta f - |\nabla f|^2 + R) + f - n] (4\pi\tau)^{-\frac{n}{2}} e^{-f}$

$\mathcal{W}(g(t), f(t), \tau(t)) = \int_{M^n} \psi \, d\mu$

$D^*\psi = -2\tau \left| Ric + \nabla\nabla f - \frac{g}{2\tau} \right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f}$.

Eg. #4.



$L(\gamma) = \int_0^\tau \sqrt{\sigma} \left(\frac{|\dot\gamma|^2}{4\sigma} + R \right)^2 d\sigma$

$\ell(x,\tau) = \frac{1}{2\sqrt{\tau}} \inf_\gamma \{L(\gamma)\}$

$\tilde{V}(t) = \int_{M^n} v \, d\mu \nearrow$ in t.

$\psi = v = (4\pi\tau)^{-n/2} e^{-\ell(x,\tau)}$.

$D^*\psi \le 0$

(R^n, δ), $\quad \mathrm{grad}(-\frac{1}{4}|x|^2)$.

$$r(\tau) = \sqrt{\delta/\tau}\ \ X$$

Eg #5. $\psi = v$ as before.

$$\varphi = R - R_{min}(0) > 0.$$

$$\int_{M^n} [R - R_{min}(0)]\, v\, d\mu \nearrow \text{ in } t.$$

$$-kg \le Rc \le kg$$

$$e^{-2kF}\ \frac{d_0^2(\bar{x}, x)}{4\tau} - C_n k\tau \le l(x,\tau) \le e^{2kF}\ \frac{d_0^2(\bar{x}, x)}{4\tau} + C_n k\tau$$

$\varphi : M^n \times [\alpha, \omega) \to R$ smooth.

$\Psi : M^n \times [\alpha, \omega) \to R$. positive.

$$\psi_r = \log(r^n \Psi), \quad \psi \equiv \psi_1.$$

$$E_r = \{(x,t) : \psi_r > 0\}$$

Assumption #1: Ψ is a locally Lipschitz.

#2: $\exists\ \Omega$, Ψ bounded outside $\Omega \times [\alpha, \omega)$.

#3: $\exists\ \bar{r} > 0$, $|\nabla\psi| \in L^2(E_{\bar{r}})$

$$\lim_{s \to \infty} \int_{E_r \cap (M^n \times \{s\})} |\psi|\, d\mu = 0.$$

$$P_{\varphi, \Psi}(r) = \int_{E_r} \{|D\psi|^2 - \psi_r \operatorname{tr}_g h\}\, \varphi\, d\mu\, dt$$

Thm: $0 < r_0 < r_1 \le \bar{r}_t$

(1): If Ψ is smooth, $\frac{D^t \Psi}{\Psi} \in L^1(E_{\bar{r}})$, then

$$\frac{P(r_1)}{r_1^n} - \frac{P(r_0)}{r_0^n} = \int_{r_0}^{r_1} \frac{n}{r^{n+1}} \int_{E_r} \left[\frac{D^t \Psi}{\Psi} - \psi_r D\varphi\right] d\mu dt\, dr$$

(2): If Ψ is locally Lipschitz.

$$\frac{P(r_1)}{r_1^n} - \frac{P(r_0)}{r_0^n} \le -\int_{r_0}^{r_1} \psi_r D\varphi\ d\mu dt\, dr.$$

Eg # 1': $(\mathbb{R}^n, \delta)$, $h \equiv 0$.

$\Psi(x,t)$ as before.

$D\Psi = 0$.

$$p(r) = \int_{E_r(y,s)} \varphi(x,t) \frac{|y-x|^2}{4(s-t)^2}\, dx\, dt$$

$$\varphi(y,s) = \frac{1}{r^n} \int_{E_r} \varphi(x,t) \frac{|y-x|^2}{4(s-t)^2}\, dx\, dt.$$

Pini (1951): $n=1$.

Eg #2', Watson (73). $n>1$:

$$\frac{p(r)}{r^n} \nearrow \text{ in } r.$$

$\varphi \equiv 1$. $\Theta_{MCF} = \lim_{r \downarrow 0} \frac{p(r)}{r^n}$

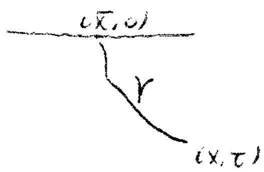
Eg 3': Corollary: $(M^n, g(\tau))$, $\frac{\partial}{\partial t} g = 2h$

Ψ = conjugate heat kernel, φ: smooth.

$$\varphi(\bar{x}, \bar{t}) = \lim_{r \downarrow 0} \frac{p(r)}{r^n} \longrightarrow$$

$$\varphi(\bar{x}, \bar{t}) = p(r_1)/r_1^n + \int_0^{r_1} \frac{n}{r^{n+1}} \int_{E_r} \Psi_r D\varphi \, du\, dt\, dr$$

Eg. 4': Ψ: reduced volume density.



$$p(r) = \int_{E_r} \left\{ |\nabla \ell|^2 + R \left(n \log \frac{r}{\sqrt{4\pi\tau}} - \ell \right) \varphi \right\} du\, dt$$

$$\varphi(x, 0) = \lim_{r \to 0} \frac{p(r)}{r^n} \Rightarrow$$

$$\varphi(\bar{x}, 0) \geq \frac{p(r)}{r^n} + \int_0^{r_1} \frac{n}{r^{n+1}} \int_{E_r} \Psi_r D\varphi \, du\, dr.$$

$R \geq 0$, on E_r.

$\Rightarrow$: $R(\bar{x}, 0) \geq \frac{1}{r_1^n} \int_{E_{r_1}} \{ |\nabla \psi|^2 + R \psi_r \} R \, du\, dt$

$+ \int_0^1 \frac{2n}{r^{n+1}} \int_{E_{r_1}} \psi_r |Ric|^2 \, du\, dt\, dr.$

In a compact shrinking soliton, Q = parabolic vector.

$\Theta_{RF}(t) = \hat{V}_0(t) = \frac{\rho_0(r)}{r^n} \quad (\varphi = 1).$

$E_r(t) = E_r \cap (M^n \times \{t\})$

$E_r(t)$, $E_r(t)$ "empty"

$\nu = - \frac{1}{|D\psi|} \nabla \psi$

$\int_{E_r} \{ (D\varphi) \psi - \varphi D^* \psi \} \, du\, dt + \int_{\partial E_r} \varphi |D\psi| \, d\sigma\, dt = 0.$

$I(r) := \int_{E_r} \{ |\nabla \psi|^2 - tr g(h) \psi_r \} \varphi \, du\, dt.$

$\frac{d}{dr} \int_{E_r} \{ |\nabla \psi|^2 \varphi \} \, du\, dt = \frac{n}{r} \int_{\partial E_r} |\nabla \psi| \varphi \, d\sigma\, dt$

$\frac{d}{dr} \int_{E_r} (tr g h) \psi_r \varphi \, du\, dt = \frac{n}{r} \int_{E_r} (tr g h) \varphi \, du\, dt$

$I(r) - \frac{r}{n} I'(r) = \int_{E_r} (\psi_r D\varphi - \frac{D^* \Psi}{\Psi} \varphi) \, du\, dt$

$p(r) = \int_{E_r} \{ |\nabla \log \Psi|^2 - tr g h) \log r^n \Psi \} \varphi \, du\, dt.$