Geometric flows on asymptotically flat manifolds

Todd A. Oliynyk

Max-Planck-Institut für Gravitationsphysik
Albert-Einstein-Institut
Golm, Germany

MOTIVATION

Action: worldsheet nonlinear sigma model

$$S(x) = \frac{1}{4\pi\alpha^{'}} \int_{\Sigma} \gamma^{\alpha\beta} g_{ab}(x) \partial_\alpha x^a \partial_\beta x^b dV(\gamma) + \frac{1}{4\pi\alpha^{'}} \int_{\Sigma} x^* B$$
$$+ \frac{1}{2\pi} \int_{\Sigma} R(\gamma) \Psi(x) dV(\gamma)$$

where

- (Σ, γ) is a 2-dimensional Riemannian manifold
- (M, g) is a n-dimensional Riemannian manifold (called the *target space*)
- $x : \Sigma \to M$; $(\theta^\sigma) \mapsto (x^1(\theta), \ldots, x^n(\theta))$ is a map,
- R_γ is the Ricci scalar of γ,
- $B = \frac{1}{2} B_{ab} dx^a \wedge dx^b$ is a two form on M,
- Ψ is a scalar field on M, and
- α' is the string coupling constant.

Cutoffs and Renormalization Group flow

- To (perturbatively) quantize the classical theory a momentum cutoff $\Lambda > 0$ must be introduced.

- This gives rises to a one parameter family of quantum field theories

$$\{ \overbrace{x^1_\Lambda, \ldots, x^n_\Lambda}^{\text{Quantum Fields}} \;,\; \overbrace{g^\Lambda_{ab}, B^\Lambda_{ab}, \Psi^\Lambda}^{\text{“coupling constants”}} \}$$

- Varying Λ induces a flow on the coupling constants

$$\{g^\Lambda_{ab}, B^\Lambda_{ab}, \Psi^\Lambda\}$$

Introducing $\tau = -\ln(\Lambda)$ gives

$$\partial_\tau g_{ab} = -\beta^g_{ab} := -\alpha^{'}\Big(R_{ab} + 2\nabla_a\nabla_b\Psi - \frac{1}{4}H_{acd}H_b{}^{cd}\Big) + O(\alpha'^2) \ ,$$

$$\partial_\tau \Psi = -\beta^\Psi := -c_0 + \frac{\alpha^{'}}{2}\big(\Delta\Psi - 2|\nabla\Psi|^2 + \frac{1}{12}|H|^2\big) + O(\alpha'^2) \ ,$$

$$\partial_\tau B_{ab} = -\beta^B_{ab} := \alpha^{'}\big(\frac{1}{2}\nabla^c H_{cab} - H_{cab}\nabla^c\Psi\big) + O(\alpha^2)$$

where $H := dB$.

Remarks:

- On length scales L that satisfy

$$L \gg \max\{1/\Lambda_1, 1/\Lambda_2\}$$

  the two field theories

$$\{x^a_{\Lambda_\nu}, g^{\Lambda_\nu}_{ab}, B^{\Lambda_\nu}_{ab}, \Psi^{\Lambda_\nu}\} \quad (\nu = 1, 2)$$

  are physically equivalent.

- This distinguishes two limits:

  IR: $\Lambda \to 0 \Longleftrightarrow \tau \to \infty$

  - This corresponds to the large distance limit.

  UV: $\Lambda \to \infty \Longleftrightarrow \tau \to -\infty$

  - This corresponds to cutoff removal.
  - For a QFT to exist as a continuum field theory it should have a good UV limit.

- Best Case Scenario: RG flow originates from a UV fixed point at $\tau = -\infty$ and flows towards an IR fixed point at $\tau = +\infty$.

Relations to Ricci flow:

- Setting the B field to zero, the beta functions to 2nd order are given by

$$\beta^g_{ab} = \alpha^{'}\big(R_{ab} + 2\nabla_a\nabla_b\Psi\big) + \frac{\alpha^{'2}}{2}R_{acde}R_b{}^{cde} + \mathrm{O}(\alpha^{'3}) \ ,$$

$$\beta^{\Psi} = c_0 - \frac{\alpha^{'}}{2}\big(\Delta\Psi - 2|\nabla\Psi|^2\big) + \frac{\alpha^{'2}}{16}R_{abcd}R^{abcd} + \mathrm{O}(\alpha^{'3})$$

- **Truncations:**
  - 1st order flow:

$$\partial_\tau g_{ab} = -\alpha^{'} R_{ab}$$

  - 2nd order flow:

$$\partial_\tau g_{ab} = -\alpha' R_{ab} - \frac{\alpha^{'2}}{2}R_{acde}R_b{}^{cde}$$

Questions of interest

- Find/classify solutions to the 1$^{\text{st}}$ order flow (i.e. Ricci flow when $B = 0$) that have
  - a UV limit (i.e. exist on an interval $(-\infty, a]$ and have a limit as $\tau \to -\infty$)
  - a IR limit (i.e. exist on an interval $[b, \infty)$ and have a limit $\tau \to \infty$))
  - both a UV and IR limit (i.e. exist on $(-\infty, \infty)$)
- Compare the difference between the 1$^{\text{st}}$ and 2$^{\text{nd}}$ order flows with the same initial data.
  - When do solutions of the 2$^{\text{nd}}$ order flow converge to solutions of the 1$^{\text{st}}$ in the limit α' goes to zero?
  - What happen to solutions to the 2$^{\text{nd}}$ order flow near critical blow up times τ_c of the 1$^{\text{st}}$ order flow?
- Find entropies for any of the truncated flows or if possible for the full RG flow.
  - These entropies can be used to rule out periodic solutions and limit cycles which are problematic for the physics.

Applications of Perelman type entropies

Entropy for the 1^{st} order flow

Define

$$F[g,B,\psi] = \int_M \left(4|\nabla u|^2 + \left(R - \frac{1}{12}|H|^2\right)u^2 \right) dV$$

and

$$\lambda[g,B] = \inf_{\{u \in H^1 | \|u\|_{L^2}=1\}} F[g,B,u] \,.$$

[Ref: T. Oliynyk, V. Suneeta, and E. Woolgar, Nucl. Phys. B **739** (2006), 441]

Proposition *The* 1^{st} *order RG flow for* (g, B)*, with diffeomorphism gauge fixed to be a solution* P *of*

$$R - \frac{1}{12}|H|^2 + 2\Delta P - |\nabla P|^2 = \lambda[g, B] \ ,$$

is the gradient flow generated by the potential $\lambda[g, B]$*:*

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} g_{ij} \\ B_{ij} \end{pmatrix} &\equiv \begin{pmatrix} -\alpha'\left(R_{ij} + \nabla_i\nabla_j P - \frac{1}{4}H_{ikl}{H_j}^{kl}\right) \\ \frac{\alpha'}{2}\left[\nabla^k H_{kij} - H_{kij}\nabla^k P\right] \end{pmatrix} \\ &= \alpha' \text{ Grad } \lambda[g, B] \ , \end{aligned} \qquad (1)$$

and $\lambda(t)$ *is monotone increasing along the gradient flow:*

$$\begin{aligned} \frac{d\lambda}{dt} &= \alpha' \int_M \left[\left| R_{ij} + \nabla_i\nabla_j P - \frac{1}{4}H_{ikl}{H_j}^{kl} \right|^2 \right. \\ &\qquad \left. + \frac{1}{4}\left|\nabla^k H_{kij} - H_{kij}\nabla^k P\right|^2 \right] e^{-P} dV \end{aligned}$$

Furthermore, fixed points of (1) (where $H = dB$*) are stationary points of* λ*.*

Tseytlin's proposed potential

[Ref: A.A. Tseytlin, preprint [hep-th/0612296]]

$$\begin{aligned}\mathcal{S}(g) &:= \widehat{S}(g,\varphi)\\ &:= \int_M \left[\bar{\beta}^{\phi} - \frac{1}{4} g^{ij}\bar{\beta}^g_{ij}\right]_{\phi=\varphi} e^{-2\varphi} dV + \lambda \left(\int_M e^{-2\varphi} dV - 1 \right) ,\end{aligned}$$

where λ is a Lagrange multiplier, c_0 is a free parameter, $\varphi := \phi[g]$ solves the eigenvalue problem

$$\begin{aligned}&\alpha' \left(\Delta - \frac{1}{4}R - \frac{\alpha'}{16}|\mathrm{Riem}|^2 + \mathcal{O}(\alpha'^2) \right) \Phi = -(\lambda + c_0)\Phi ,\\ &\int_M \Phi^2 dV = 1 ,\\ &\varphi = -\log \Phi ,\end{aligned}$$

and as before

$$\begin{aligned}\bar{\beta}^g_{ij} &= \alpha' (R_{ij} + 2\nabla_i \nabla_j \phi) + \frac{\alpha'^2}{2} R_{iklm} R_j{}^{klm} + \mathcal{O}(\alpha'^3) ,\\ \bar{\beta}^{\phi} &= c_0 - \alpha' \left(\frac{1}{2}\Delta\phi + |\nabla\phi|^2 \right) + \frac{\alpha'^2}{16}|\mathrm{Riem}|^2 + \mathcal{O}(\alpha'^3) .\end{aligned}$$

and the flow is

$$\partial_\tau g_{ij} = -\bar{\beta}^g_{ij}[\varphi] = -\alpha' (R_{ij} + 2\nabla_i \nabla_j \varphi) - \frac{\alpha'^2}{2} R_{iklm} R_j{}^{klm} + \mathcal{O}(\alpha'^3) ,$$

From this we get

$$\frac{d\widehat{S}}{dt} = -\frac{1}{4}\int_M dV \ e^{-2\varphi}\left[|\beta^g_{ij}|^2 - \frac{\alpha'}{2}|\nabla_k\beta^g_{ij} - \nabla_i\beta^g_{kj}|^2\right] + \mathcal{O}(\alpha'^3|\beta^g|) \ .$$

A problem on AF manifolds

[Ref: T. Oliynyk, V. Suneeta, and E. Woolgar, Phys. Lett. B.**610** (2005), 115]

- To construct Perelman's entropy, we need to solve
$$-4\Delta u + Ru = \lambda u$$
for $u > 0$ where λ is the lowest eigenvalue.

- If $R \geq 0$, then
$$\lambda = \inf_{\{u \in H^1 | \|u\|_{L^2}=1\}} \int_M 4|\nabla u|^2 + Ru^2 dV = 0$$

- If $R(x_0) < 0$ for some $x_0 \in M$, then there always exists a constant $\kappa \geq 1$ large enough such that
$$\lambda = \inf_{\{u \in H^1 | \|u\|_{L^2}=1\}} \int_M 4|\nabla u|^2 + \kappa Ru^2 dV < 0$$

Weighted Sobolev Spaces

Ref: T.A. Oliynyk and E. Woolgar, *Asymptotically flat Ricci flows*, preprint [math.DG/0607438]

Weighted L^p norms

$$\|u\|_{L^p_\delta} := \begin{cases} \|\sigma^{-\delta-n/p}\, u\|_{L^p(\mathbb{R}^n)} & \text{if } 1 \le p < \infty \\ \|\sigma^{-\delta}\, u\|_{L^\infty(\mathbb{R}^n)} & \text{if } p = \infty \end{cases}$$

where $\sigma(x) := \sqrt{1+|x|^2}$

Weighted Sobolev norms

$$\|u\|_{W^{k,p}_\delta} := \begin{cases} \Big(\sum_{|I|\le k} \|D^I u\|^p_{L^p_{\delta-|I|}}\Big)^{1/p} & \text{if } 1 \le p < \infty \\ \sum_{|I|\le k} \|D^I u\|_{L^\infty_{\delta-|I|}} & \text{if } p = \infty \end{cases}$$

where $D^I = \partial^{I_1}_{x^1} \dots \partial^{I_n}_{x^n}$ and $|I| = \sum_{i=1}^n I_i$

Calculus Inequalities

Sobolev

$$\|u\|_{L^\infty_\delta} \le C\|u\|_{W^{k,p}_\delta} \quad (\text{if } n - kp < 0)$$

Multiplication

$$\|uv\|_{W^{k_3,p}_{\delta_3}} \le C\|u\|_{W^{k_1,p}_{\delta_2}}\|v\|_{W^{k_2,p}_{\delta_1}}$$

wherever $k_1, k_2 \ge k_3$, $\delta_1 + \delta_2 \le \delta_3$, and $k_3 < k_1 + k_2 - n/p$.

Moser

$$\|[D^I, u]v\|_{L^2_{\delta-|I|}} \le C\|\nabla u\|_{H^{k-1}_{\delta_1}}\|v\|_{L^\infty_{\delta_4}} + C\|\nabla u\|_{L^\infty_{\delta_2}}\|v\|_{H^{k-1}_{\delta_3}}$$

provided $|I| \le k$ and $\delta = \delta_1 + \delta_4 = \delta_2 + \delta_3$

Quasilinear parabolic equations

$$\partial_t u = a^{ij}(v,u)\partial_i\partial_j u + b(v,u,\nabla u) + f,$$
$$u|_{t=0} = u_0$$

where

(i) the maps $u = u(t,x)$ and $f = f(t,x)$ are $\mathbb{R}^m$-valued while $v = v(t,x)$ is a $\mathbb{R}^r$-valued,

(ii) $a^{ij} \in C^\infty(\mathbb{R}^{m+r}, \mathbb{M}_{m\times m})$, a^{ij} is symmetric for each $i,j = 1,\ldots,n$,

(iii) $b \in C^\infty(\mathbb{R}^{r+m(1+n)}, \mathbb{M}_{m\times m})$, $b(0) = 0$, and

(iv) there exists a constant $\omega > 0$ such that

$$(w|a^{ij}(u,v)\xi_i\xi_j \cdot w) \geq \omega|\xi|^2|w|^2$$

for all $u,w \in \mathbb{R}^m$, $v \in \mathbb{R}^r$ and $\xi \in \mathbb{R}^n$.

Theorem 1. *Suppose, $\delta \le 0$, $\ell \ge k > n/2+1$, $u_0 \in H^k_\delta$, and $v, f \in C^0([0,T], H^k_\delta)$ for some $T > 0$. Then there exists a $T_* \in (0,T)$ and a $u \in L^\infty((0,T_*), H^k_\delta) \cap \mathrm{Lip}([0,T_*), H^{k-2}_\delta)$ that solves the initial value problem*

$$\partial_t u = a^{ij}(v,u)\partial_i\partial_j u + b(v,u,\nabla u) + f,$$
$$u|_{t=0} = u_0.$$

Theorem 2. *The solution $u \in L^\infty([0,T_*], H^k_\delta) \cap \mathrm{Lip}([0,T_*], H^{k-2}_\delta)$ from theorem 1 is unique and satisfies the additional regularity*

$$u \in C^0([0,T_*), H^k_\delta) \cap C^1([0,T_*), H^{k-2}_\delta)$$

and

$$u \in C^0([T_1,T_2], H^\ell_\delta) \cap C^1([T_1,T_2], H^{\ell-2}_\delta)$$

for every closed interval $[T_1,T_2] \subset (0,T_*)$. Moreover, if $\sup_{0\le t<T_*} \|u(t)\|_{W^{1,\infty}} < \infty$ then there exist a $T^* \in (T_*,T)$ such u can be extended to a solution on $[0,T^*)$.

Asymptotically flat manifolds

We assume

- $M \cong \mathbb{R}^n$
- $(x^1, \ldots, x^n)$ are Cartesian coordinates
- $\partial_i = \dfrac{\partial}{\partial x^i}$

The manifold M along with a complete Riemannian metric

$$g = g_{ij} dx^i dx^j \qquad g^{ij} g_{jk} = \delta^i_k$$

is *asymptotically flat of class* H^k_δ provided

$$\delta < 0 \,, \quad k > n/2 \,,$$

and

$$g_{ij} - \delta_{ij} \,,\; g^{ij} - \delta^{ij} \in H^k_\delta .$$

Local Existence

Theorem 3. *Let $\hat{g}$ be an asymptotically flat metric of class H^k_δ with $\delta < 0$ and $k > n/2 + 3$. Then there exists a $T > 0$ and a family $\{g(t), t \in [0, T)\}$ of asymptotically flat metrics of class H^{k-2}_δ such that $g(0) = \hat{g}$,*

$$g_{ij} - \delta_{ij}\,,\ g^{ij} - \delta^{ij} \in C^1([0,T), H^{k-2}_\delta)\,,$$

and $\partial_t g_{ij} = -2R_{ij}$ for all $t \in [0,T)$. Moreover, $g(t,x) \in C^\infty((0,T) \times M)$ and $g_{ij} - \delta_{ij}$, $g^{ij} - \delta^{ij} \in C^1([T_1, T_2], H^\ell_\delta)$ for any $\ell \geq 0$ and $0 < T_1 < T_2 < T$.

Ricci-DeTurck flow

$$\partial_t g_{ij} = -2R_{ij} + \nabla_i W_i + \nabla_j W_j \quad : \qquad g(0) = \hat{g}$$
$$W_j = g_{jk} W^k := g_{jk} g^{pq} (\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq})$$

$$\begin{aligned}\partial_t h_{ij} = g^{ij}\partial_i\partial_j h_{ij} + \tfrac{1}{2} g^{pq} g^{rs} \big(&\partial_i h_{pr}\partial_j h_{qs} + 2\partial_p h_{jp}\partial_q h_{is} \\ &- 2\partial_p h_{jp}\partial_s h_{iq} - 2\partial_j h_{pr}\partial_s h_{iq} - 2\partial_i h_{pr}\partial_s h_{jq}\big)\,,\\ g_{ij} = \delta_{ij} + h_{ij}\end{aligned}$$

Remark 4. Related results:

- **Existence:**
  - W Shi, J Diff Geom **30** (1989) 223
  - B List, *Evolution of an extended Ricci flow system*, PhD thesis, Max-Planck-Instituts für Gravitationsphysik und der Freien Universität Berlin, unpublished

- **Uniqueness:** BL Chen and XP Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, preprint 2005 [arxiv.org:math.DF/0505447]

- **Preservation of the asymptotics:**
  - B List, *Evolution of an extended Ricci flow system*, PhD thesis, Max-Planck-Instituts für Gravitationsphysik und der Freien Universität Berlin, unpublished
  - X Dai and L Ma, *Mass under the Ricci flow*, preprint 2005 [arxiv.org:math.DG/0510083]

Corollary 5. *Let $k > n/2 + 4$ and $g(t)$ be the Ricci flow solution from Theorem 3. Then $R_{ij} \in C^1([0, T), H^{k-4}_{\delta-2})$ and $g_{ij}(t) = \widehat{g}_{ij} + f_{ij}(t)$ where $f_{ij} \in C^1([0, T), H^{k-4}_{\delta-2})$. Moreover, if $k > n/2 + 6$, $\delta < 4 - n$ and $\widehat{R} \in L^1$ then $R(t) \in C^1([0, T), L^1)$.*

Remark 6. *The mass of an asymptotically flat metric g of class $H^k_\delta \subset W^{2,2n/(n-2)}_\delta$ ($k \geq 3$) is well defined and given by the formula*

$$\textit{mass}(g) := \int_{S_\infty} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) dS^i$$

provided $\delta \leq (2 - n)/2$ and the Ricci scalar is both non-negative and integrable. In this case it follows easily that

$$\textit{mass}(g(t)) = \textit{mass}(\widehat{g}) \quad \text{for all } t \geq 0.$$

Uniqueness

Theorem 7. *Suppose* $k > n/2 + 4$, $\delta < 0$, *and* $\tilde{g}(t)$ *and* $\bar{g}(t)$ *are two solutions to the Ricci flow satisfying* $\bar{g}(0) = \tilde{g}(0)$ *and*

$$\tilde{g}_{ij} - \delta_{ij}\,,\ \tilde{g}^{ij} - \delta^{ij}\,,\ \bar{g}_{ij} - \delta_{ij}\,, \bar{g}^{ij} - \delta^{ij} \in C^1([0,T), H^k_\delta).$$

Then $\bar{g}(t) = \tilde{g}(t)$ *for all* $t \in [0,T)$.

Continuation

Theorem 8. *Suppose* $k > n/2 + 4$, $\delta < 0$ *and* $\hat{g}$ *is an asymptotically flat metric of class* H^k_δ. *Then Ricci flow* $\partial_t g_{ij} = -2R_{ij}$ *with the initial condition* $g(0) = \hat{g}$ *has a unique solution on a maximal time interval* $0 \le t < T_M \le \infty$. *If* $T_M < \infty$ *then*

$$\limsup_{t \to T_M} \sup_{x \in \mathbb{R}^n} |\mathrm{Rm}(t,x)|_{g(t,x)} = \infty\,.$$

Moreover, for any $T \in [0, T_M)$,
$K = \sup_{0 \le t \le T} \sup_{x \in \mathbb{R}^n} |\mathrm{Rm}(t,x)|_{g(t,x)} < \infty$ *and*

$$e^{-2KT}\hat{g} \le g(t) \le e^{2KT}\hat{g} \quad \textit{for all } t \in [0,T].$$

Rotationally symmetric solutions

Ricci flow in area radius coordinates

We consider Ricci-DeTurck flow

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \nabla_i \xi_j + \nabla_j \xi_i \ ,$$

with

$$g(t) = f^2(t,r)dr^2 + g_{\mathsf{can}}$$

and

$$\xi = \xi_1(t,r)dr$$

where

$$\xi_1 = \left[\frac{(n-2)}{r}\left(f^2(t,r)-1\right) + \frac{\frac{\partial f}{\partial r}}{f(t,r)} \right] \ .$$

Then $f(t,r)$ satisfies

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{f^2}\frac{\partial^2 f}{\partial r^2} - \frac{2}{f^3}\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{(n-2)}{r} - \frac{1}{rf^2}\right)\frac{\partial f}{\partial r} \\ &\quad - \frac{(n-2)}{r^2 f}\left(f^2-1\right) \ . \end{aligned}$$

A continuation principle

Let

$$\lambda_1(t,r) = \frac{1}{rf^3(t,r)} \frac{\partial f(t,r)}{\partial r}$$

and

$$\lambda_2(t,r) = \frac{1}{r^2}\left(1 - \frac{1}{f^2(t,r)}\right).$$

These are the sectional curvatures in planes containing and orthogonal to dr, respectively.

Theorem 9. *If there exists a constant $C_\lambda > 0$ independent of T_M such that*

$$\sup_{0<r<\infty} \big(|\lambda_1(t,r)| + |\lambda_2(t,r)|\big) \le C_\lambda \,,$$

then $T_M = \infty$.

Type III rotationally symmetric solutions

Theorem 10. *Let* $\{x^i\}_{i=1}^n$ *be a fixed Cartesian coordinate system on* $\mathbb{R}^n$, $n \geq 3$. *Let* $\widehat{g} = \widehat{g}_{ij} dx^i dx^j$ *be an asymptotically flat, rotationally symmetric metric on* $\mathbb{R}^n$ *of class* H^k_δ *with* $k > n/2 + 4$ *and* $\delta < 0$. *If* $(\mathbb{R}^n, \widehat{g})$ *does not contain any minimal hyperspheres, then there exists a solution* $g(t,x) \in C^\infty((0,\infty) \times \mathbb{R}^n)$ *to Ricci flow such that*

(i) $g(0,x) = \widehat{g}(x)$,

(ii) $g_{ij} - \delta_{ij} \in C^1([0,T], H^{k-2}_\delta)$ *and* $g_{ij} - \delta_{ij} \in C^1([T_1,T_2], H^\ell_\delta)$ *for any* $0 < T_1 < T_2 < \infty$, $0 < T < \infty$, $\ell \geq 0$,

(iii) for each integer $\ell \geq 0$ *there exists a constant* $C_\ell > 0$ *such that*

$$\sup_{x \in \mathbb{R}^n} |\nabla^\ell \mathrm{Rm}(t,x)|_{g(t,x)} \leq \frac{C_\ell}{(1+t)t^{\ell/2}} \quad \forall\, t > 0\,,$$

(iv) the flow converges to n*-dimensional Euclidean space* $\mathbb{E}^n$ *in the pointed Cheeger-Gromov sense as* $t \to \infty$, *and*

(v) if furthermore $k > n/2 + 6$, $\delta < \min\{4-n, 1-n/2\}$, $\widehat{R} \geq 0$, *and* $\widehat{R} \in L^1$, *then the ADM mass of* $g(t)$ *is well defined and* $\mathrm{mass}(g(t)) = \mathrm{mass}(\widehat{g})$ *for all* $t \geq 0$.

Quasi-local mass

- Brown-York quasi-local mass contained within a closed hypersurface Σ:

$$\mu[\Sigma] := \int_\Sigma (H_0 - H) d\Sigma$$

where H is the mean curvature of Σ and H_0 is the mean curvature of the image of Σ under an isometric embedding of Σ into flat space (assuming there is such an embedding).

- If Σ is a hypersphere of radius $r = b(t)$, then

$$\mu(t) \;=\; b^{n-2}(t)\left(1 - \frac{1}{f(t,b(t))}\right)\text{vol}\left(\mathbb{S}^{n-1}, \text{can}\right) \; .$$

The three most interesting kinds of hyperspheres are those of

(i) fixed surface area

$$b(t) = b_0 = const > 0 \ ,$$

(ii) fixed volume contained within

$$\int_0^{b(t)} \int_{\mathbb{S}^{n-1}} f(t,r) r^{n-1} dr d\Omega =: V_0 = const > 0 \ , \text{and}$$

(iii) fixed proper radius

$$\int_0^{b(t)} f(t,r) dr =: R_0 = const > 0 \ .$$

In all three of these cases the Brown-York mass behaves as

$$\mu(t) \sim 1/t \quad \text{as} \quad t \to \infty \, .$$

Maximum Principles

Let

$$w(t,r) := f^2(t,r) - 1\,.$$

Then w satisfies

$$\frac{\partial w}{\partial t} = \frac{1}{f^2}\frac{\partial^2 w}{\partial r^2} - \frac{3}{2f^4}\left[\frac{\partial w}{\partial r}\right]^2 + \left[\frac{n-2}{r} - \frac{1}{rf^2}\right]\frac{\partial w}{\partial r} - \frac{2(n-2)}{r^2} w\,.$$

This can be used to determine upper and lower bounds on w and hence f in terms of the initial data, i.e.

Proposition 11. *Define constants $C^{\pm}_{f^2}$ such that $0 < C^{-}_{f^2} := \min_{r\in[0,\infty)}\{f^2(0,r)\}$ and let $C^{+}_{f^2} := \max_{r\in[0,\infty)}\{f^2(0,r)\}$. Then*

$$0 < C^{-}_{f^2} \le f^2(t,r) \le C^{+}_{f^2}\;.$$

for all $(t,r) \in D = [0,T_M) \times [0,\infty)$.

For $m \in (0,2]$, define

$$u_m(t,r) := \left(\frac{1+t}{r^m + r^2}\right)\left(\frac{1}{f^2(t,r)} - 1\right) = -\left(\frac{1+t}{r^{m-2}+1}\right)\lambda_2(t,r)$$

Then

$$\begin{aligned}
\frac{\partial u_m}{\partial t} \quad = \quad & \frac{1}{f^2}\frac{\partial^2 u_m}{\partial r^2} - \frac{(r^m + r^2)}{2(1+t)}\left(\frac{\partial u_m}{\partial r}\right)^2 - \frac{(2r + mr^{m-1})}{(1+t)} u_m \frac{\partial u_m}{\partial r} \\
& + \left[\frac{2\left(2r + mr^{m-1}\right)}{(r^m + r^2) f^2} - \frac{1}{rf^2} + \frac{(n-2)}{r}\right]\frac{\partial u_m}{\partial r} \\
& - \frac{(2-m)(m+n-2)}{r^2(1+r^{2-m})} u_m \\
& + \frac{1}{(1+t)}\left\{\frac{1}{1+r^{2-m}}\left[u_m - ((4-m)(m+n-2) + m(n-2))\, u_m^2\right]\right. \\
& + \frac{r^{2-m}}{1+r^{2-m}}\left[u_m - 2(n-1) u_m^2\right] \\
& \left. + \frac{r^m}{1+r^{2-m}}\left[(m-2)(m+n-2) - m\left(\frac{m}{2} + n - 2\right)\right] u_m^2\right\} .
\end{aligned}$$

Proposition 12. *There is a constant C_u^+ which depends only on the initial data $f(0,r)$ such that $u_m(t,r) \le C_u^+$ for all $t \in [0, T_M)$ and all $m \in (0,2)$.*

For $m \in (0,2]$, define

$$v_m(t,r) := \left(\frac{1+t}{r^m + r^2}\right)\left(f^2(t,r) - 1\right) = \left(\frac{1+t}{r^{m-2}+1}\right) f(t,r)\lambda_2(t,r)\,.$$

Then

$$\begin{aligned}
\frac{\partial v_m}{\partial t} \quad = \quad & \frac{1}{f^2}\frac{\partial^2 v_m}{\partial r^2} - \frac{3(r^m + r^2)}{2f^4(1+t)}\left(\frac{\partial v_m}{\partial r}\right)^2 \\
& + \frac{2(2-m)r}{f^2}\left[\frac{1}{r^m + r^2} - \frac{3v_m}{2f^2(1+t)}\right]\frac{\partial v_m}{\partial r} \\
& + (m-2)\left[\frac{(m-1)}{f^2(r^m+r^2)} + \frac{1}{r^2}\left(n - 2 + \frac{m}{f^2}\right)\right] v_m \\
& + \frac{1}{(1+t)}\left[v_m - \frac{3}{2f^2}\left(\frac{(m-2)^2}{1+r^{m-2}} + m^2\left(r^{m-2}+1\right)\right) v_m^2\right] \ .
\end{aligned}$$

Proposition 13. *There is a constant C_v^+ which depends only on the initial data $f(0,r)$ such that $v_m(t,r) < C_v^+$ for all $(t,r) \in [0, T_M) \times [0,\infty)$ and all $m \in (1,2)$.*

For $m \in (1,2]$ define

$$y_m(t,r) := \left(\frac{1+t}{1+r^{2-m}}\right)\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r^m}\left(\frac{1}{f}-1\right)\right]\right\} = \frac{(1+t)f}{r^{m-2}+1}\left(\frac{m}{(1+f)}\lambda_2-\lambda_1\right)$$

Then

$$\begin{aligned}\frac{\partial y_m}{\partial t} &= \frac{1}{f^2}\frac{\partial^2 y_m}{\partial r^2}+\frac{1}{r}\alpha_m\frac{\partial y_m}{\partial r}\\ &+\frac{1}{r^2}\left\{\left[\frac{2}{f}\left(2(m-1)r^m+mr^2\right)y_m+1\right]\frac{y_m}{1+t}\right.\\ &\left.+\beta_m y_m+(1+t)\gamma_m\right\},\end{aligned}$$

where

$$\begin{aligned}\alpha_m &:= \frac{2(r^m+r^2)}{f}\frac{y_m}{(1+t)}+\frac{4m-3}{f^2}-\frac{2m}{f}+n-2\\ &-\frac{2(m-2)r^{2-m}}{f^2(1+r^{2-m})},\\ \beta_m &:= \frac{7m^2-14m+4}{f^2}-\frac{m(6m-8)}{f}+(n-2)\left(m-1-\frac{3}{f^2}\right)\\ &+\frac{(m-2)r^{2-m}}{1+r^{2-m}}\left[-\frac{(3m-2)}{f^2}+\frac{2m}{f}-(n-2)\right],\\ \gamma_m &:= \frac{1}{(r^m+r^2)}\left(\frac{1}{f}-1\right)\left\{\frac{2m(m-1)(m-2)}{f^2}+\frac{2m^2(2-m)}{f}\right.\\ &\left.+(n-2)\left[-m+\frac{m+2}{f}+\frac{2(1-m)}{f^2}\right]\right\}.\end{aligned}$$

Proposition 14. *Let $\frac{3}{2}\le m<2$. There exists a constant C_y^- which depends only on the initial data $f(0,r)$ such that $y_m(t,r)>C_y^-$ for all $(t,r)\in[0,T_M)\times[0,\infty)$ and $m\in(3/2,2)$.*

Let

$$\tilde{R} := (1+t)R \ .$$

Then

$$\frac{\partial \tilde{R}}{\partial t} \geq \Delta \tilde{R} + \xi \cdot \nabla \tilde{R} + \frac{1}{(1+t)}\left(\frac{2}{n}\tilde{R}^2 + \tilde{R}\right).$$

Proposition 15. *If R is the scalar curvature of a Ricci flow developing from asymptotically flat initial data on a manifold M then there is a constant $C_R^- \leq 0$ such that on $[0, T_M) \times [0, \infty) \ni (t, r)$ we have*

$$R \geq \frac{C_R^-}{1+t} \ . \tag{2}$$

Corollary 16. *Then $\lambda_1(t, r)$ is bounded below on $[0, T_M) \times [0, \infty) \ni (t, r)$ by*

$$\lambda_1(t, r) \geq \frac{1}{(1+t)}\left(\frac{C_R^-}{2(n-1)} - \frac{(n-2)C_v^+}{C_{f^2}^-}\right) \ .$$

Questions under investigation

- Are the spherically symmetric solutions stable under non-spherically symmetric perturbations?
  - For a certain class of asymptotically flat metrics, it has already been established by Schnürer, Schulze and Simon that the flat metric is stable under Ricci flow for small enough perturbations.
- When minimal hyperspheres are present what conditions on the initial data guarantee global existence or blow up?
  - On S^n this has been previously investigated: S Angenent and D Knopf, Math. Res. Lett. 11 (2004), 493.
- Does adding the B field make any fundamental changes?
  - In his thesis, Bernhard List studies a related, but simpler type of flow where the two-form B is effectively replaced by a scalar function u. This could be a good place to start.

- What effect does the adding the curvature term $\frac{\alpha'^2}{2} R_{acde} {R_b}^{cde}$ to Ricci flow have?
  - On $\mathbb{T}^n$ using maximal regularity theory, Christine Guenther has been able to show that the flat metric continues to be stable under small enough perturbations.

- Other asymptotics?