Note Taker Checklist Form -*MSRI*

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Talk Title and Workshop assigned to:
Differential Harnack inequalities and monotone quantities

Lecturer (Full name): Reto Müeller
Date & Time of Event: 3-15-07. 3:30pm - 4:30pm

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: Stationary manifold. Ricci flow
   Differential Harnack inequality

2. Please summarize the lecture in 5 or less sentences.
   He writes too soft/gentle to read.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

3-15-07. 3:30 - 4:30 by Reto Müeller.

Part (I): Stat. Manifold

M^n closed Riemannian manifold. $\square = \partial_t - \Delta = D$

Prop (I). (Li-Yau, Hamilton).

$0 < u \in C^\infty(M \times [0,T])$, $\square u = 0$

(1): $Rc \geq 0$

$$H = \partial_t u - \frac{|\nabla u|^2}{u} + \frac{nu}{2t} \geq 0 \quad \text{on } M \times [0,T]$$

(2): $\nabla Rc \equiv 0$. $(Rc(V,W)\,W,\,V) \geq 0$, $\forall V, W \in \Gamma(TM)$

$$H_{ij} = \nabla_i \nabla_j u - \frac{\nabla_i u \nabla_j u}{u} + \frac{n}{2t} g_{ij} \geq 0 \quad \text{on } M \times [0,T]$$

Pf: (1): $L_{ij} = H_{ij} - \frac{n}{2t} g_{ij}$

$$(*)\left\{ \begin{aligned} \partial_t L_{ij} &\overset{=\Delta L_{ij}}{\cancel{}} \ \frac{2}{u} L_{ij}^2 + 2R_{kijl} L_{kl} - R_{ik} L_{jk} - R_{jk} L_{ik} \\ &+ \frac{2}{n} R_{kijl} \nabla_k u \nabla_l u + (\nabla_p R_{ij} - \nabla_i R_{jp}) \nabla_p u \end{aligned}\right.$$

$$\partial_t H_{ij} = \partial_t L_{ij} + \frac{\Delta u}{2t} g_{ij} - \frac{u}{2t^2} g_{ij}$$

$$\geq \Delta H_{ij} + \cdots$$

(2): $L = H - \frac{un}{2t}$

$(*)$.
$$L^2 \leq n\,|L_{ij}|^2$$

$$\Rightarrow \partial_t L \geq \Delta L + \frac{2}{nu} L^2 + \frac{2}{n} Rc(\nabla_i \nabla_j)$$

$$\Rightarrow \partial_t H \geq \Delta H + \cdots + \frac{2}{n} \underbrace{Rc(\nabla u_i \nabla u_j)}_{\geq 0}$$

$$\partial_t(\log u) = \frac{\partial_t u}{u} \overset{LY}{\geq} \frac{|\nabla u|^2}{u^2} - \frac{n}{2t} = |\nabla \ln u|^2 - \frac{n}{2t} \qquad (**)$$

on a path $\gamma(t)$

$$\frac{\partial}{\partial t}\log u(\gamma(t),t) = \partial_t \log u + \nabla \log u \cdot \dot\gamma$$

$$\overset{(**)}{\geq} \left(\nabla \log u + \frac{\dot\gamma}{2}\right)^2 - \frac{1}{4}|\dot\gamma|^2 - \frac{n}{2t}$$

$$\log \frac{u(x_2,t_2)}{u(x_1,t_1)} = \log u(\gamma(t),t)\Big|_{p_1}^{p_2}$$

$$\geq \int_{t_1}^{t_2} \left(-\frac{1}{4}|\dot\gamma|^2 - \frac{n}{2t}\right) dt$$

$$= -\frac{n}{2}\log\left(\frac{t_2}{t_1}\right) - \int_{t_1}^{t_2} \frac{1}{4}|\dot\gamma|^2\, dt$$

Cor: If $Ric \geq 0$, $0 \leq u$, $\nabla u = 0$ then

$$u(x_2,t_2) \geq u(x_1,t_1)\left(\frac{t_2}{t_1}\right)^{-\frac{n}{2}} e^{\ell(x_1,x_2,t_1,t_2)}$$

where

$$\ell(x_1,x_2,t_1,t_2) = \inf_{\gamma\in\Gamma} \frac{1}{2(t_1-t_2)}\int_0^1 \frac{1}{2}|\dot\gamma|^2 dt$$

with $\Gamma: \gamma[0,1] \to M \mid \gamma(0)=x_1,\ \gamma(1)=x_2)$

$$u = (4\pi t)^{-\frac{n}{2}} e^{-f}.$$

L-Y: $\left(-\Delta f + \frac{n}{2t}\right)u \geq 0 \iff f(2\Delta f) - n \leq 0$

on M with $Ric \geq 0$

$$f(x,t) \leq \frac{d(p,x)^2}{4t} \quad \text{if } u \to \delta_0\ (t \downarrow 0)$$

Prop. I: $W(u,t) = t(2\Delta f - |\nabla f|^2) + f - n \leq 0$.

$$\text{if } u \to \delta_p\ (t \downarrow 0), \quad \square u = 0.$$

$$0 \le -\frac{1}{2t} W(u,t) = -\Delta f + \frac{n}{2}|\nabla f|^2 + \frac{n-2}{2t} \geq \partial_t f - \frac{1}{2}|\nabla f|^2 + \frac{n}{2t}$$

$$\frac{\partial}{\partial t} f(r(t),t) = \partial_t f + \langle \nabla f, \dot{r} \rangle$$

$$\le \partial_t f + \frac{1}{3}|\nabla f|^2 + \frac{3}{2}$$

$$\le \frac{1}{2}|\dot{g}|^2 - \frac{f}{2t}.$$

$$\frac{d}{dt}(2\sqrt{t} f) \le \sqrt{t}\,|\dot{g}|^2.$$

$$\left| 2\sqrt{T}\, f(q,T) - 2\sqrt{t}\,(f(r(t)),t) \right|_0^T$$

$$\le \int_0^T \sqrt{t}\,|\dot{r}|^2 dt.$$

Corollary: let u be the heat kernel on M with $Ric \geq 0$.

and let p be the point where the δ-function for $t \downarrow 0$.

then

$$u(q,T) \geq (4\pi T)^{-\frac{T}{2}} e^{-l(n,T)}$$

where

$$l(n,T) = \inf_{r \in \Gamma} \left(\frac{1}{2\sqrt{T}} \int_0^T \sqrt{t}\,|\dot{g}|^2 dt \right)$$

where

$$\Gamma := \{ r : [0,1] \to M \mid r(0) = p,\ r(1) = q \}$$

Prop 3: (i): $|\nabla l(q,T)| = \frac{1}{T} l(q,T)$

(ii): $\partial_T l(q,T) = -\frac{1}{T} l(q,T)$

(iii): $\Delta l(q,T) \le \frac{n}{2T} - \frac{1}{2T^{1/2}} \int_0^T 2t^{1/2}\, Ric(\dot{\imath},\dot{\jmath})\, dt$.

thus: $\partial_t - \Delta l + |\nabla l|^2 + \frac{n}{2T} \geq 0$. if $Ric \geq 0$

i.e. $(4\pi t)^{-\frac{n}{2}} e^{l(n,T)} = v(q,T)$ satisfies $\nabla v \le 0$.

If $Ric \geq 0$, then the reduced volume

$$\hat{V}(t) = \int_M v(x,t)\, d\mu(g) \quad \text{is non-increasing in } t.$$

证: $\partial_t \hat{V}(t) = \int_M \partial_t v(g,f)\, d\mu$

$$\leq \int_M \Delta v \, d\mu = 0.$$

New Entropy

$$N(u) = \int u \ln u \, dV$$

$$\hat{N}(u,t) = 4\left[N(u) + \frac{n}{2}\log(4\pi\tau) + \frac{n}{2}\right]$$

Prop 4:

If $\Box u = 0$, $u = e^{-f}$.

(i) $\exists\ \partial_t N = -\int_M |\nabla f|^2 e^{-f}\, d\mu = F(u) \geq 0$.

(ii) $\partial_t F = \int_{t_1}^{t_2} 2n\, |\nabla f|^2 + Ric(\nabla_i f\, \nabla_j f)\Big]\, d\mu$

证: (Nei Li).

If $\nabla u = 0$,

(i) $\partial_t \hat{N} = -W(t)$.

(ii) $\partial_t W = -\int 2nt\left(|\nabla f - \frac{g}{2t}\right)^2 + Ric(\nabla f_i\, \nabla f) \quad d\mu$

Part (II): (Ricci flow).

$\partial_t g = -2Ric$; M: closed. $\square^* = \partial_t - \Delta + R$.

Prop 1. (Hamilton)

$g_{ij}(t)$ weakly positive curvature operator.

(i): $H(V) = \partial_t R + \frac{R}{t} - 2\nabla_i R V_j + 2R_{ij} V_i V_j \geq 0$

(ii): $H(W, V) = \cdots \geq 0$. (matrix version).

$g(t)$ is a good steady soliton if $g(t) = \varphi_t^*(g(0))$,

$\varphi_t : M \to M$. diffeo. generated by $X = \nabla f$.

$g(t)$ is a grad. shrinker if $g(t) = (T-t)\varphi_t^*(g(0))$.

φ_t as above.

Lemma: (i) if $g(t)$ is grad. steady, $R_{ij} + \nabla_i \nabla_j f = 0$

$\Leftrightarrow$ Moreover, $u = e^{-f}$ when $\square^* u = 0$

(ii). $g(t)$. g_{ij} shrink

$\Leftrightarrow R_{ij} + \nabla_i \nabla_j f - \frac{g_{ij}}{2T} < 0$.

Moreover,

$u = (4\pi\tau)^{-\frac{n}{2}} e^{-f} \Leftrightarrow \square^* u = 0$.

do the similar applications as above.

Part III: (Bernhard's List's flow).

$$\begin{cases} \partial_t g_{ij} = -2R_{ij} + 4\nabla_i\varphi\nabla_j\varphi \\ \partial_t\varphi = \Delta\varphi \end{cases}$$

$\varphi: M \to R$.

$R_{ij} = 2\nabla_i\varphi\nabla_j\varphi$.

$\Delta\varphi = 0$.

φ: Lapse function.

$S_{ij} = R_{ij} - 2\nabla_i\varphi\nabla_j\varphi$

$S = R - 2|\nabla\varphi|^2$.

Can also get similar monotone quantities as above.