MSRI
April 16, 07
MMP workshop

Extension Thms

Thm (H-McK)

X = sm proj var

$T+\Delta$ = eff $\mathbb{Q}$-div. on X w SNC support

- T irred, $[\Delta]=0$
- $\Delta \equiv A+B$, A ample, B eff, $T \not\subseteq \operatorname{Supp} B$
- $\mathbb{B}(K_X+T+\Delta)$ does not contain any intersectn of comps of $T+\Delta$

Choose k s.t. $k\Delta$ integral, set

$$L = k(K_X+T+\Delta)$$

Then

$$H^0(X, mL) \twoheadrightarrow H^0(T, mL_T)$$

for all $m \geq 0$.

Rmk: H-McK prove analogous assertion for proj $X \to$ (affine)

History. Siu, Kawamata, Takayama, H-McK, ...

I. Multiplier & Adjoint Ideals

- For eff $\mathbb{Q}$-div D on X, have

$$\mathcal{J}(X, D) \subseteq \mathcal{O}_X :$$

measures sings of D,
more sings $\longleftrightarrow$ deeper D

$\left(\begin{array}{l} \mu : X' \longrightarrow X \text{ log resln} \\ \mathcal{J}(D) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [\mu^* D]) \end{array} \right)$

- Let

$$L = \text{big lb on } X.$$

Set

$$\mathcal{J}(X, \|L\|) = \mathcal{J}\left(\tfrac{1}{p} D_p\right)$$

for $D_p \in |pL|$ general, $p \gg 0$.

Prop:

(i). $$H^0(L \otimes \mathcal{J}(\|L\|)) \xrightarrow{\ \cong\ } H^0(L)$$

Def. Given $\mathfrak{a} \subseteq \mathcal{O}_X$, ~~locating~~ $s \in \Gamma(X, M)$ say s vanishes on $\mathfrak{a}$ $\Leftrightarrow$ $s \in \operatorname{im}(H^0(M \otimes \mathfrak{a}) \to H^0(M))$

(ii) If $M = K_X + L + P$ w P nef, then

$$H^i(X, M \otimes \mathcal{J}(\|L\|)) = 0 \quad i > 0$$

(iii) If $M = K_X + L + P$ w $P = (\dim + 1)$ (very ample)
then

$$\mathcal{O}(M) \otimes \mathcal{J}(\|L\|) \text{ glob gen.}$$

(Pf: (ii) + C-M Regularity)

- Now consider smooth divisor

$$T \subseteq X, \qquad T \not\subseteq \operatorname{Supp} D.$$

Can ~~define~~ constr <u>adjoint ideal</u>

$$\operatorname{Adj}_T(X, D) \subseteq \mathcal{O}_X$$

$$\left(\begin{array}{l} \mu : X' \to X \\ \mu^* T = T' + R \\ \operatorname{Adj} = \mu_*(K_{X'/X} - [\mu^* D] - R) \end{array} \right)$$

w.

$$0 \to \mathcal{J}(X, D) \otimes \mathcal{O}(-T) \to \operatorname{Adj} \longrightarrow \mathcal{J}(T, D_T) \to 0$$

- Assuming $\mathbb{B}(L) \not\supseteq T$, have

$$\mathcal{J}(T, \|L_T\|) \supseteq \mathcal{J}(T, \|L\|_T) = \mathcal{J}(T, \tfrac{1}{p} D_p | T)$$

$D_p \in |pL|$ on X

- Gives $\operatorname{Adj}_T(X, \|L\|) \subseteq \mathcal{O}_X$, w

$$0 \to \mathcal{J}(X, \|L\|) \otimes \mathcal{O}(-T) \longrightarrow \operatorname{Adj} \longrightarrow \mathcal{J}(T, \|L\|_T) \to 0$$

Idea: $\otimes M$

<u>Vocab.</u> Given

$$\mathfrak{a} \subseteq \mathcal{O}_X, \quad s \in \Gamma(X, M)$$

say that s <u>vanishes along</u> $\mathfrak{a}$ if

$$s \in \operatorname{Im}\left(H^0(M \otimes \mathfrak{a}) \hookrightarrow H^0(M) \right).$$

<u>So.</u> If $s \in \Gamma(T, M_T)$ vanishes along $\mathcal{J}(T, \|L\|_T)$, and $M - (K_X + L + T)$ is nef, then s extends to X.

Main Lemma & Pf of Thm?

(II). Consider as above.

$$L = k(K_X + T + \Delta)$$
$$\Delta = A + B, \dots$$

Want: $H^0(mL) \twoheadrightarrow H^0(T, mL_T)$

Main Lemma: $\exists$ very amp H s.t. $\forall\, p \geq 0$, every

$$\sigma \in \Gamma\left(T, \mathcal{O}(pL_T + H_T) \otimes \mathcal{J}(\|pL_T\|)\right)$$

extends to $\hat{\sigma} \in \Gamma(X, \mathcal{O}(pL + H))$.

Grant for now

Pf of Thm:

(0). Can assume (T, B_T) is KLT. Take $h \in \Gamma(X, H)$ gen.

(1°) Let

$$s \in \Gamma(T, mL_T), \qquad \text{(Want to extend to } X\text{)}$$

consider

$$s^\ell \cdot h \in \Gamma(T, \ell m L_T + H_T) \qquad (\ell \gg 0)$$

$$s^\ell h \text{ vanishes on } \mathcal{J}(T, \|m\ell L_T\|),$$

so

$\exists$

$$\hat{\sigma} \in \Gamma(X, m\ell L + H)$$

s.t.

$$\hat{\sigma}|T = s^\ell h.$$

(2°). Let

$$F = \frac{mk-1}{m\ell k}\,\mathrm{div}(\hat{\sigma}) + B \qquad \text{eff } \mathbb{Q}\text{-div on } X$$

Compute:

$$F \equiv (m-1)L + (k-1)(K_X + T + \Delta) + B + \left(\frac{mk-1}{mk\ell} H\right)$$

So

$$mL - F - T \equiv K_X + \Delta - B - \left(\frac{mk-1}{m\ell k} H\right)$$

$$= K_X + \text{ample} \quad \text{for } \ell >> 0$$

(3°) Using (T, B_T) KLT, ℓ gen, see

$$\mathcal{O}_T(-\mathrm{div}(s)) \subseteq \mathcal{J}\left(T, \mathrm{div}(s) + B_T + \frac{mk-1}{m\ell k} H_T\right)$$

$$\subseteq \mathcal{J}(T, F_T).$$

So s vanishes on $\mathcal{J}(T, F_T)$.

(4°). Consider

$$0 \to \mathcal{J}(X,F)\otimes\mathcal{O}(-T) \longrightarrow \mathrm{Adj} \longrightarrow \mathcal{J}(T,F_T) \longrightarrow 0$$

$$\otimes \qquad\qquad \otimes \qquad\qquad \otimes$$

$$\mathcal{O}_X(mL) \qquad\qquad \mathcal{O}_X(mL) \qquad\qquad \mathcal{O}_T(mL_T)$$

Then

$$s \in H^0(T, \text{term on right})$$

$$H^1(X, \text{term on left}) = 0$$

So s extends!

(III). Proof of Main Lemma —

- Case $k = 1$: Assume

$$L = K_X + T \qquad (\Delta = 0 : \text{don't need } \Delta = A + B)$$

$$T \not\subseteq \mathbb{B}(L). \quad \underline{\text{NB}} \quad L_T = K_T$$

Will prove

$(*)_p$ | If $H = (\dim X + 1)(\text{very ample})$ then $H^0\Big(T, \ (pL_T + H_T) \otimes \mathcal{J}(\|(p-1)L_T\|)\Big)$ lifts to $\Gamma(X, (pL + H))$.

- Induction on p, $p = 1$ OK.
- Assume $(*)_p$ holds.

(1°). Claim:

$$\mathcal{J}(T, \|(p-1)L_T\|) \subseteq \mathcal{J}(T, \|pL+H\|_T)$$

Pf: · CM Reg & Van $\Longrightarrow$

$$\mathcal{O}_T(pL_T + H_T) \otimes \mathcal{J}(T, \|(p-1)L_T\|) \qquad (**)$$

is glob gen

· $(*)_p \Rightarrow$ sectns of sheaf in $(*)$ lift, so:

$$\mathcal{J}(T, \|(p-1)L_T\|) \subseteq \mathfrak{b}(X, |pL+H|) \cdot \mathcal{O}_T$$

Claim follows

(2°). Consider adj ex seq $\otimes \mathcal{O}_X((p+1)L+H)$

$$0 \longrightarrow \begin{matrix} \mathcal{J}(\|pL+H\|) \otimes \mathcal{O}(-T) \\ \otimes \\ (p+1)L+H \end{matrix} \longrightarrow \begin{matrix} \mathrm{Adj} \\ \otimes \\ (p+1)L+H \end{matrix} \longrightarrow \begin{matrix} \mathcal{J}(T, \|pL+H\|_T) \\ \otimes \\ (p+1)L_T+H_T \end{matrix} \longrightarrow 0$$