Notes on Birkar-Cascini-Hacon-M^c^Kernan

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Assumption: We proved flip in dimension n, using minimal models in dimension $n-1$.

Main Theorem: Minimal models in dim. n.

Restriction: Δ is a big $\mathbb{R}$-divisor and $K+\Delta$ is (pseudo) effective.

Corollary 1. *Δ is a* **big** *$\mathbb{Q}$-divisor, (X,Δ) is klt and $K+\Delta$ is* **effective**. *Then the canonical ring*

$$\sum_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)) \quad \text{is finitely gen.}$$

Proof. Get minimal model for (X,Δ), then use base point freeness. □

Corollary 2. *X smooth, projective and K_X is* **big**. *Then the canonical ring*

$$\sum_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X)) \quad \text{is finitely generated.}$$

Proof. Pick some effective $D \sim_{\mathbb{Q}} mK_X$. Then $(X,\epsilon D)$ is klt (even terminal) for $0<\epsilon \ll 1$ and ϵD is big. So $(X,\epsilon D)$ has a minimal model. It is automatically a minimal model for X. □

Corollary 3. *X smooth, projective. Then the canonical ring*

$$\sum_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X)) \quad \textit{is finitely generated.}$$

Proof. As in Kodaira's canonical bundle formula for elliptic surfaces, Fujino-Mori reduces the ring to a general type situation in lower dimension. □

Corollary 4. *Let X be a Fano variety. Then the Cox ring*

$$\sum_{D\in \mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(D)) \quad \textit{is finitely generated.}$$

Proof. See original. □

Corollary 5. *If $K+\Delta$ not pseudo effective, then $\exists\ X \dashrightarrow X'$ and a Mori fiber space $X' \to Z'$.*

Proof. See original. □

Comparison of 3 MMP's

Start		Goal
(X, Δ)	$\dashrightarrow$	$(X, \Delta)^{min}$

Mori-MMP (libertarian): Pick **any** extremal ray, contract/flip as needed. Eventually we get $(X, \Delta)^{min}$.

MMP with scaling (dictatorial): Fix H and $t_0 > 0$ such that $K + \Delta + t_0 H$ is nef.
Let $t \to 0$. Critical value: $\exists$ extremal ray R_1 st.
$K + \Delta + t_1 H$ is nef but
$R_1 \cdot \big(K + \Delta + (t_1 - \eta)H\big) < 0$ for $\eta > 0$.
Contract/flip this R_1.

Roundabout MMP of BCHM

1. Increase Δ in a mild manner to Δ^+.
2. Increase Δ^+ wildly to $\Delta^+ + M$.
3. Construct $(X, \Delta^+ + M)^{min}$.
4. Remove the excess M to get $(X, \Delta^+)^{min}$.
5. Note that $(X, \Delta^+)^{min}$ is also $(X, \Delta)^{min}$.

$(X, \Delta) \xrightarrow{1,2} (X, \Delta^+ + M) \xrightarrow{3} (X, \Delta^+ + M)^{min} \xrightarrow{4} (X, \Delta^+)^{min}$

"bending it like BCHM"

Spiraling induction

$\boxed{\text{MMP with scaling in dim. } n-1}$

$\Downarrow$

$\boxed{\text{Termination with scaling in dim. } n \text{ near } \lfloor \Delta \rfloor}$

$\Downarrow$ "bending"

$\boxed{\text{Existence of } (X,\Delta)^{min} \text{ in dim. } n}$

$\Downarrow$

$\boxed{\text{Finiteness of } (X,\Delta+\sum t_iD_i)^{min} \text{ in dim. } n \text{ for } 0 \le t_i \le 1}$

$\Downarrow$

$\boxed{\text{MMP with scaling in dim. } n}$

Note: Finiteness can fail if Δ is not big.

Side issues:
1. Δ klt/dlt/lc
2. $K+\Delta$ limit of effective $\Rightarrow$ effective.

MMP with scaling

1. Start: $K + \Delta + t_0 H$ nef, $t_0 > 0$.
2. Set $t = t_0$ and decrease it.
3. We hit a first *critical value* $t_0 \geq t_1$:
   - $K + \Delta + t_1 H$ nef but
   - $K + \Delta + (t_1 - \eta)H$ is not nef for $\eta > 0$.

$\Rightarrow \exists$ extremal ray (or face) $R \subset \overline{NE}(X)$ such that

$$R \cdot (K + \Delta + t_1 H) = 0 \quad \text{and} \quad R \cdot H > 0.$$

Thus: $R \cdot (K + \Delta) < 0$ and R is a "usual" extremal ray (or face).

Contract/flip $X_0 \dashrightarrow X_1$ and continue.

Problem: What if we get an infinite sequence

$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots$$

Advantage of scaling: Each X_i is a minimal model for some $K + \Delta + tH$.

$\Rightarrow$ If we know finiteness of models as t varies, there is no infinite sequence.

$$\boxed{\text{Finiteness of } (X, \Delta + tH)^{min} \text{ in dim. } n \text{ for } 0 \leq t \leq t_0}$$

$$\Downarrow$$

$$\boxed{\text{MMP with scaling in dim. } n}$$

Note: In Mori-MMP, the X_i are *not* minimal models of anything predictable.

MMP with scaling near $\lfloor \Delta \rfloor$

Start with $(X, S+\Delta)$, S integral.
Get a series of contractions/flips

$$\cdots (X_i, S_i + \Delta_i) \overset{\phi_i}{\dashrightarrow} (X_{i+1}, S_{i+1} + \Delta_{i+1}) \cdots$$

Claim: Only finitely many i s.t. $S_i \cap \mathrm{Ex}(\phi_i) \neq \emptyset$.

Proof. (Old knowledge:)

1. the discrepancy $a(E, X_i, S_i + \Delta_i)$
   – weakly increases with i and
   – strictly increases if $\mathrm{center}_{X_i}(E) \subset \mathrm{Ex}(\phi_i)$.
2. Very few E with $a(E, X_i, S_i + \Delta_i) < 0$.
3. If $S_i \dashrightarrow S_{i+1}$ creates a *new* divisor $F_{i+1} \subset S_{i+1}$, then $\exists\, E_{i+1}$ with $a(E_{i+1}, X_{i+1}, S_{i+1} + \Delta_{i+1}) \leq 0$.
4. Thus: no new divisors for $i \gg 1$.
5. Only finitely many $S_i \dashrightarrow S_{i+1}$ contracts a divisor (Picard number of S_i drops).

Corollary. For $i \gg 1$, each $S_i \dashrightarrow S_{i+1}$ is an isomorphism in codimension 1 (= *traverse*).

So: If $X_0 \dashrightarrow X_1 \cdots$ is MMP with scaling, then

$$S_N \dashrightarrow S_{N+1} \dashrightarrow S_{N+1} \dashrightarrow \cdots$$

is MMP with scaling (plus isomorphisms).

$$\boxed{\text{MMP with scaling in dim. } n-1}$$
$$\Downarrow$$
$$\boxed{\text{Termination with scaling in dim. } n \text{ near } \lfloor \Delta \rfloor}$$

Assume: Termination with scaling near $\lfloor \Delta \rfloor$

Scaling to boundary. Assume that

- $K + \Delta \sim cH + F$ for some $c \geq 0$ and $F \geq 0$
- $K + \Delta + H$ is nef, and
- $\mathrm{Supp}(F) \subset \lfloor \Delta \rfloor$

$\Rightarrow (X, \Delta + t \cdot H)^{min}$ exists for every $0 \leq t \leq 1$.

Proof. Start scaling. At critical value, get a ray R such that $R \cdot H > 0$ and $R \cdot (K + \Delta) < 0$.
Thus $R \cdot (cH + F) < 0$ and $R \cdot F < 0$.
$\Rightarrow$ locus$(R) \subset \mathrm{Supp}(F) \subset \lfloor \Delta \rfloor$. □

Need: large $\lfloor \Delta \rfloor$

Useless divisor lemma.
If $\Delta' \subset$ stable base locus of $(K + \Delta)$, then
$(X, \Delta + \Delta')^{min} = (X, \Delta)^{min}$.

Proof. Note that

$$(\Delta')^{min} \subset \text{stable base locus of } (K+\Delta)^{min} = \text{stable base locus of } (K + \Delta + \Delta')^{min}$$

However, $(K + \Delta + \Delta')^{min}$ is base point free,
thus $(\Delta')^{min} = 0$, and so
$(X, \Delta + \Delta')^{min} = (X, \Delta)^{min}$. □

Bending I. $K + \Delta$ is a $\mathbb{Q}$-divisor

0. $K + \Delta \sim_{\mathbb{Q}} rM + F$, M mobile, irreducible and F is in stable base locus.

1. Take log resolution. Add all new divisors to F. Increase Δ to get $\mathrm{Supp}(F^+) \subset \lfloor \Delta^+ \rfloor$.

2. Add M to Δ^+ and pick ample H to get

- $K + \Delta^+ + M \sim 0 \cdot H + \big((r+1)M + F^+\big)$
- $K + \Delta^+ + M + H$ is nef, and
- $\mathrm{Supp}(M + F^+) \subset \lfloor \Delta^+ + M \rfloor$.

3. Scale by H to get $(X, \Delta^+ + M)^{min}$:

- $K + \Delta^+ \sim rM + F^+$
- $K + \Delta^+ + M$ is nef, and
- $\mathrm{Supp}(F^+) \subset \lfloor \Delta^+ \rfloor$.

4. Scale by M to get $(X, \Delta^+)^{min}$.

5. By useless divisor lemma ($\Delta' := \Delta^+ - \Delta$):
$$(X, \Delta^+)^{min} = (X, \Delta)^{min}.$$

Bending II. $K + \Delta$ is an $\mathbb{R}$-divisor

0. $K + \Delta \sim_{\mathbb{Q}} r_i M_i + F$, where M_i mobile and
   F in stable base locus (not obvious).

1–3 goes as before to get

- $K + \Delta^+ \sim \sum r_i M_i + F^+$ $\quad K+\Delta^+ \sim r_1 M_1 + (\sum_{i=2} r_i M_i + F^+)$
- $K + \Delta^+ + \sum M_i$ is nef, and $\quad K + (\Delta^+ + \sum_{i=2}^{n} M_i) + M_1$
- $\mathrm{Supp}(F^+) \subset \lfloor \Delta^+ \rfloor$.

4.(first try): scale $\sum M_i$.
   Problem: $\sum r_i M_i \neq c \sum M_i$.

(second try): scale M_1. This works, but
   Problem: support condition fails at next step
      when we scale M_2.

Solution: Do not take *all* of M_1 away at the
      at the first step.

4. (third try) reorder: $r_1 \le r_2 \le \cdots \le r_k$.

4.1.j. For $j = 1, \dots, k$ construct minmod. for

$$\big(X, \Delta^+ + \frac{1}{r_j}(r_1 M_1 + \cdots + r_{j-1} M_{j-1}) + \sum_{i \ge j} M_i\big)$$

Proof of $j \to j+1$:
Move M_j to the left to get

$$\big(X, \Delta^+ + \frac{1}{r_j}(r_1 M_1 + \cdots + r_j M_j) + \sum_{i \ge j+1} M_i\big)$$

Scale $r_1 M_1 + \cdots + r_j M_j$ with
scale factor r_j / r_{j+1}. This gives

$$\big(X, \Delta^+ + \frac{1}{r_{j+1}}(r_1 M_1 + \cdots + r_j M_j) + \sum_{i \ge j+1} M_i\big)$$

After step 4.1.k:

- $K + \Delta^+ \sim \sum r_i M_i + F^+$
- $K + \Delta^+ + \frac{1}{r_k}(r_1 M_1 + \cdots + r_{k-1} M_{k-1}) + M_k =$
  $K + \Delta^+ + \frac{1}{r_k}(r_1 M_1 + \cdots + r_k M_k)$ is nef.
- $\mathrm{Supp}(F^+) \subset \lfloor \Delta^+ \rfloor$.

4.2. Last step: Scale by $r_1 M_1 + \cdots + r_k M_k$
to get $(X, \Delta^+)^{min}$.

5. By useless divisor lemma
$(X, \Delta^+)^{min} = (X, \Delta)^{min}$.

Finiteness of models

By compactness, enough to prove:

Finiteness of $(X, \Delta' + \sum t_i D_i)^{min}$ for $|t_i| \leq \epsilon(\Delta')$

Note: need this for $\mathbb{R}$-divisors Δ'!

Proof. Induction on r for $D_1, \ldots, D_r$.
Let $(X^m, \Delta^m) := (X, \Delta')^{min}$. Base point freeness:
$g : X^m \to X^c$ such that $K_{X^m} + \Delta^m \sim g^*(\text{ample})$.

$\Rightarrow$ for $|t_i| \ll 1$, $g^*(\text{ample}) \gg \sum t_i D_i$,
except on the fibers of g.

$\Rightarrow$ MMP to get $(X^m, \Delta^m + \sum t_i D_i)^{min}$ is relative to X^c.

Working over X^c, we can assume that $K_{X^m} + \Delta^m \sim 0$. Thus:

$$K_{X^m} + \Delta^m + c \sum t_i D_i \sim c\big(K_{X^m} + \Delta^m + \sum t_i D_i\big)$$

$$\Rightarrow (X^m, \Delta^m + \sum t_i D_i)^{min} = (X^m, \Delta^m + c \sum t_i D_i)^{min}$$

Choose c such that $\max_i |ct_i| = \epsilon$. That is:
$(ct_1, \ldots, ct_r)$ is on a face of $[-\epsilon, \epsilon]^r$.
Faces of r-cube $= 2r$ copies of $(r-1)$-cube.

Done by induction on r.