Eyssidieux { ~~Analytic~~ appr. of the F.G of the canonical ring

Try and explain section 4 of Siu's paper.

1) stable vanishing order.

$X/\mathbb{C}$ smooth proj. variety, $L$ big line bundle on $X$.

$m \in \mathbb{N}_{>0}$. $S_1^{(m)}, \cdots, S_{N_m}^{(m)}$ basis of $H^0(mL)$

$\Psi_m = \sum_{j=1}^{N_m} |S_j|^{\frac{2}{m}}$ $\|$ smooth metric on $L$.

Prop: $h_m = 1$. $\Psi_m^{-1}$ a $\geq 0$ singular Hermitian metric on $X \overset{def}{\Longleftrightarrow}$ (H) $\geq 0$ closed positive (1,1)-current

$(\varepsilon_m)_{m \in \mathbb{N}}$ to be a sequence of $>0$ number s.t

$\sum \varepsilon_m \max_X |\Psi_m| < \infty$

$\Phi = \sum_{m=1}^{\infty} \varepsilon_m \Psi_m$. $\Phi_m = \sum_{h=1}^{\infty} \varepsilon_m \Psi_m$

Prop. $|\ .\ |\Phi^{-1}$ and $|\ .\ |\Phi_m^{-1}$ are $\geq 0$ singular Hermitian metric $\Phi \geq \Phi_m$

Def: $\cdot P \in X$, the vanishing order of $\Phi$ is precisely achieved at $P$ iff

$\exists m \gg 1$ $\exists U \ni P$ open nghb : $\exists C > 0$ s.t $\Phi_m \geq C\Phi$. on $U$

$\cdot$ the vanishing order of $\Phi$ is precisely achieved in codim $1$, $\exists Z \subset X$. $\operatorname{codim} Z \geq 1$

and the vanish order of $\Phi$ is achieved precisely at every pt $P \notin Z$

Remark(1). $\Phi$ precisely achieves its vanishing order at $P$

$\forall m \gg 1$, $\gamma(\varphi_m, P) = \gamma(\varphi, P)$ $\quad \varphi_m = \log \Phi_m$

stable vanishing order at $P$

$$\gamma(\varphi_m, P) = 2 \inf_{\substack{D_k \in |kL| \\ k \leq m}} \frac{\operatorname{mult}_P(D_k)}{k}$$

(2) $\bigoplus_{m \geq 0} H^0(mL)$ f.g $\Longrightarrow$ precise achievement of vanishing order.

$*$ for $L = K_X$ $\Longleftarrow$

(3) $\Phi$ doesnot achieve stable vanishing order in codim 1.

$B = \bigcap_{m \in \mathbb{N}} B_s(|m!\,L|) = Z_1 \cup \cdots \cup Z_k$ irreducible decomposition.

$Z_1, \cdots Z_s$ are precisely the codim 1 components

$\Longrightarrow \exists\ i \in \{1, \cdots, s\}$ $\forall P \in Z_i^{gen}$ $\Phi$ doesn't achieve precisely its vanishing order at $P$.

$\Updownarrow$

$\exists\ i \in \{1, \cdots, s\}$ $\forall m \in \mathbb{Z}_{>0}$, $\forall s \in H^0(mL)$

$\operatorname{div}(s) = m\eta Z_i + F$ where $Z_i \not\subset V(F)$, $\eta > \frac{\gamma(\log \Phi, P)}{2}$

Furthermore, the following holds $\exists m_k \uparrow \infty$, $S_K \in H^0(m_K L)$

$$\eta_K = \frac{\mathrm{mult}_{Z_i}(S_K)}{mk} \qquad \eta_K \xrightarrow[K\to\infty]{} \frac{r(\phi, P)}{2}$$

II. $L = K_X$. $X$ is general type.

Assume the vanishing order of $\phi$ is not achieved in codim 1.

i.e. $\exists M \subseteq X$, codim $M=1$, $\exists$ $S_K \in |m_K K_X|$

$\frac{1}{m_K}\mathrm{mult}_{Z_i}(S_K) \downarrow \gamma_i$, $\gamma_i = \frac{\text{generic Lelong number of } \log\phi \text{ along } Z_K}{2}$

and $S \in |mK_X|$ $\frac{1}{m}$ mult $> \gamma$

(1) construction of 2 singular metrics

step 1 assume $\gamma \in \mathbb{Q}$

choose $m_1 \in \mathbb{N}$, $m_1\gamma \in \mathbb{N}$

$L = m_1(K_X - \gamma M)$ $\qquad S_M \in \Gamma(\mathcal{O}(M))$

$\widetilde{\Phi} = \frac{\Phi^{m_1}}{|S_M|^{2\gamma m_1}}$ $\qquad \widetilde{\varphi} = \log\widetilde{\Phi}$

lemma: $\widetilde{\Phi}$ is a sing. metric on $L$. $r(\theta_{\widetilde{\Phi}}, P) = 0$ where $P$ generic pt of $M$

~~$\widetilde{\varphi} = \log\widetilde{\Phi}$~~

Step 2: lemma 1: For $p_0 \gg 0$, $\exists e^{-\chi}$, $a>0$ sing. metric on $p_0L - K_X$, s.t.

$\mathcal{J}_{\chi,p} = I_{M,p}$ $\forall p \in M^{gen}$

$\exists$ $e^{-\zeta}$, $a>0$. singu. metric on $p_0L - K_X$ s.t. $\mathcal{J}_{\zeta\, p} = \mathcal{O}_{X,p}$ $\forall p \in M^{gen}$

Furthermore, one may assume that $\zeta \geq \chi$ so that $\mathcal{J}_\chi \subseteq \mathcal{J}_\zeta$.

Proof: $X$ is of general type. $K_X \underset{\mathbb{Q}}{\equiv} A + E$ $A$-ample $E$ effective

$S_E^k$ be the canonical ~~(multi)~~ section of $\mathcal{O}(kE)$

$\forall m$ divisible enough. $S_E^m \cdot H(X; A_m^1) \subseteq H^0(X, mK_X)$ ($A_m^1$ very ample)

$E = \eta M + F$

$M \not\subseteq \mathrm{sup}(F)$ $\quad \eta > \gamma$. $p_0L - K_X \equiv pm_1(K_X - \gamma M) - K_X$

$$= (p_0m_1 - 1)(K_X - \gamma M) - (\gamma + 1)M + M$$

let $0 < \beta < p_0m_1 - 1 \equiv (p_0 m_1 - 1 - \beta)(K_X - \gamma M) + \beta K_X - (\gamma + 1 - \beta(\eta - \gamma))M + M$

$$\equiv \cdots\cdots\cdots + \beta A + \beta F \cdots\cdots$$

Choose $\beta = \frac{\gamma+1}{\eta-\gamma}$ $\quad p_0$ big enough

$$e^{-\chi} = \frac{e^{-\frac{-p_0 m_1 - 1 - \beta}{m_1}}\, \bar{\varphi}\, (h_A)^{\beta}}{|S_M|^2\, |S_F|^{2\beta}}$$ $e^{-\chi}$ is $>0$

$$e^{-\xi} = \frac{e^{-\frac{p_0 m_1 - 1 - \beta}{m_1}}\, \bar{\varphi}\, (h_A)^{\beta_1}}{|S_F|^{2\beta_1}}$$ $\beta_1 = \frac{\gamma}{\eta - \gamma}$

step 3. lemma $1^{st}$ also valid if $\gamma \notin \mathbb{Q}$

$m \in \mathbb{N}$. $L_m = mK_X - \lfloor m\gamma \rfloor M$

$$e^{-(p\varphi+\xi)} = \frac{e^{-p + \frac{p_0 m - 1 - \beta_1}{m_1}}\, \tilde{\varphi}\, h_A^{\beta_1}}{|S_M|^2\, |S_F|^{2\beta_1}}\, h_M^{\{m\gamma\}}$$ $\beta_1 = \frac{\gamma}{\eta - \gamma}$

singular metric on $(p+p_0)L_m - K_X$

curvature $\geq \beta_1 \omega_A - \{m\gamma\}$ curvature smooth on $M$

if $m$ is choosen so that $\{m\gamma\}$ small enough $\Theta_M > 0$

b) Application of Nadel Vanishing Thm

Corollary: $\forall s \in H^0\left(X, \frac{\mathcal{J}_{p\varphi+\xi}}{\mathcal{J}_{p\varphi+\chi}} \otimes L^{p+p_0}\right)$.

$s(Q) = 0$ where $Q \in M^{gen}$.

pf: $H^1(X, L^{p+p_0} \otimes I_{p\varphi+\chi}) = 0$ hence it lifts to $\hat{\sigma} \in \Gamma(L^{p+p_0} \otimes I)$

in case $\gamma \in \mathbb{Q}$

$\tilde{\sigma} \in (p+p_0)m_1(K_X - \gamma M)$ $\forall q \in M^{gen}$

$\Rightarrow S_M^{\sigma m_1 (p+p_0)} \tilde{\sigma}$ is a $(p+p_0)m_1$ - canonical form s.t its vanishing order at Q is $\equiv \gamma M_1$

in case $\gamma \notin \mathbb{Q}$, the analogues multiplier ideal section has vanishing order $\frac{\lfloor m\gamma \rfloor}{m}$.