Extension (Demailly)

last lecture: assume you don't have precise vanishing order along some hypersurface $M \subset X$. $s \in H^0(X, mK_X)$ vanish more than $m\gamma$

$\Theta_\Phi = \sum \gamma_j [V_j] + R$, $\Phi = \sum^{\infty} \varepsilon_m \sum_{j=1}^{N_m} |\sigma_{j,m}|^{\frac{2}{m}}$

$M = V_j$ $\quad \underbrace{K_X}_{(\text{big})} \equiv A + E$

$\gamma = \gamma_j$

what is needed is now nonvanishing Thm for such sections.

Difficulty: $\mathcal{J}_\varphi / \mathcal{J}_\psi$ ψ more sing. than φ

need not be a sheaf of modules over a reduced scheme

Basic observation. L with two metrics φ_0, φ_1 more singular than φ_0

$\varphi_t = (1-t)\varphi_0 + t\varphi_1$

$\mathcal{J}_{\varphi_t} \rightarrow$ decreases

lemma: $\mathcal{J}_{\varphi_{t_1}} / \mathcal{J}_{\varphi_{t_2}}$ $t_1 < t_2$ ~~are 2 consequence jumping pts.~~ with no jump in between

then actually define over $\mathcal{O}_X / \sqrt{\mathcal{J}_{\varphi_{t_2}}}$

Proof: Hölder's inequality

$\mathcal{J}_{\varphi_{\tau-\varepsilon}} / \mathcal{J}_{\varphi_{\tau+\varepsilon}}$ $\quad f \in \mathcal{J}_{\varphi_{\tau-\varepsilon}}$

$g \in \sqrt{\mathcal{J}_{\varphi_{\tau+\varepsilon}}} \Rightarrow fg \in \mathcal{J}_{\varphi_{\tau+\varepsilon}}$ $\quad g^p \in \mathcal{J}_{\varphi_{\tau+\varepsilon}}$

$\int |f|^2 e^{-\varphi_{\tau-\varepsilon}} < +\infty$ $\quad \int |g|^{2p} e^{-\varphi_{\tau+\varepsilon'}} < +\infty \xrightarrow{\text{Hölder}}$

Def: (L, φ) singular metric "stable" if $\mathcal{J}_{(1+\varepsilon)\varphi} = \mathcal{J}_\varphi$ for $0 < \varepsilon \ll 1$

Nonvanishing Thm: M proj. Mfd L pesudo-eff. φ stable

$\exists p_0$ $p_0 L - K_M$ metric χ big ($\Theta_\chi \geq \varepsilon\omega$) Assume $\exists p$ st

$\mathcal{J}_{p\varphi + \chi} \supset \mathcal{J}_{(p+p_0)\varphi}$

~~$(p+p_0)$~~ $(p+p_0)L - K_M$ $\quad \leftarrow (p+p_0)L$

Then for m sufficiently divisible $H^0(M, (m+p_0)L \otimes \mathcal{J}_{m\varphi+\chi}) \neq 0$ and sections generate $\mathcal{J}_{m\varphi+\chi}$ outside codim 2 in M.

pf: $\Theta_\varphi = \sum \gamma_j [V_j] + R$ a very ample divisor so that $A - K_M$ ample

s_1 section of $H^0(M, A)$ smooth Hypersurface H_1

$0 \to \mathcal{J}_{p\varphi} \otimes \mathcal{O}(pL + kA) \xrightarrow{\times s_1} \mathcal{J}_{p\varphi} \otimes \mathcal{O}(pL + (k+1)A) \to (\mathcal{J}_{p\varphi} / s_1 \mathcal{J}_{p\varphi}) \otimes \mathcal{O}(pL + (k+1)A) \to 0$

S. generic $\Rightarrow \mathcal{J}_{p\varphi}/s.\mathcal{J}_{p\varphi} = \mathcal{J}_{p\varphi}|_{H_1}$

repeat $H_1 \cap \cdots \cap H_2 \cap \cdots H_{n-1} = C$

$$h^0(M, \mathcal{J}_{p\varphi} \otimes \mathcal{O}(pL+nA)) \geq h^0(C, p\varphi|_C \otimes \mathcal{O}(pL+nA|_C))$$

$$\underset{R.R}{\geq} 1-g+\sum(p\gamma_j - \lfloor p\gamma_j \rfloor) + pR\cdot C + nA\cdot C$$

$$= 1-g+\sum\left(p\gamma_j - \lfloor p\gamma_j \rfloor\right) + pR\cdot A^{n-1} + nA^n$$

Question: is whether this $\to +\infty$

Bad case • $R \equiv 0$

• finitely many γ_j which are irrational

~~First step ($K_M \rightsquigarrow$ can't have $R \equiv 0$)~~

Good case: $h^0 \geq C_p \to +\infty$ as $p \to +\infty$

and sections from C extend to M

$\rightsquigarrow$ can get sections in $H^0(M, \mathcal{J}_{p\varphi} \otimes \mathcal{O}(pL+nA))$

which vanishes ~~at~~ high order at some pt $x_0 \in C$

$\rightsquigarrow$ metric $\mathcal{J}_{(\chi)} < \mathcal{J}_{[\frac{p}{N}]\varphi} \cdot m_{x_0}^{q}$ on $\lceil \frac{p}{N} \rceil L + \lceil \frac{nA}{N} \rceil$.

given χ. $p_0 L - K_M$

$\delta\theta + (1-\delta)\chi$ $\quad (p'+p_0)L - K_M + \varepsilon A$ write as $(p'+p_0)L - K_M$

stable

ideal sheaf not be pertubated if δ small

χ' slightly more singular than χ and especially $\mathcal{J}_{\chi'} < \mathcal{J}_{\chi} \cdot m_{x_0}^{q'}$ $\quad q' \sim q\delta$

Now proceed by induction on dimension

$H^0(M, \mathcal{J}_\chi/\mathcal{J}_{\chi'} \otimes \mathcal{O}((p+p_0)L))$ by interpolation $\mathcal{J}_\chi/\mathcal{J}_{\chi'}$ defined over reduced scheme of M.