NONVANISHING THEOREM

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The aim of this talk is to give a proof of the nonvanishing theorem in dimension n, under the assumptions of the finiteness of minimal models in dimension $n-1$, and the exsitence of minimal models in dimension n. To be clear, the theorems mean

0.1. **Theorem** (Finiteness of Minimal Models). *If X is smooth, A is ample, $A+\sum\Delta_i$ has simple normal crossing support, then the set of isomorphism classes of $\{Y \mid Y$ is minimal model of $(K_X, A+\sum_{i=1}^n t_i\Delta_i)$ where $0\le t_i\le 1\}$ is finite.*

0.2. **Theorem** (Existence of Minimal Models). *If (X,Δ) is klt, Δ is big and $K_X+\Delta\sim_{\mathbb{R}} D\ge 0$, then there exsits a minimal model for the pair (X,Δ).*

0.3. **Theorem** (Nonvanishing Theorem). *(X,Δ) is a projective klt pair, X is n-dimensional and Δ is big, if $K_X+\Delta$ is pseudo-effective, assume (0.1)$_{(n-1)}$ and (0.2)$_n$, then there exsits $D\ge 0$, such that $K_X+\Delta\sim_{\mathbb{R}} D$.*

0.4. **Remark.** The theorems here are simpler versions of the corresponding theorems in [BCHM06], and the proof in this talk is essentially one step of the spiraling induction appearing there.

Proof. Since Δ is big, we can assume $\Delta = A+B$, where A is ample and B is effective.

(step 1) If $h^0(\mathcal{O}_X(\lfloor mk(K_X+\Delta)\rfloor + kA))$ is a bounded function of m (for k sufficient divisible), then $K_X+\Delta\sim_{\mathbb{R}} D\ge 0$.

According to the work of Nakayama, we know $K_X+\Delta\equiv N_\sigma(K_X+\Delta)\ge 0$, where $N_\sigma(K_X+\Delta)$ is the limit of the stable fixed divisor of $K_X+\Delta+\epsilon A$ when $\epsilon\to 0$. Set $A'\sim_{\mathbb{R}} A+N_\sigma(K_X+\Delta)-(K_X+\Delta)$ a general ample divisor and denote $\Delta' := \Delta - A + A'\ge 0$. Then (X,Δ') is a *klt* pair satisfying $K_X+\Delta'\equiv K_X+\Delta'\sim_{\mathbb{R}} N_\sigma(K_X+\Delta)$. So by assumption, there exsits a log terminal model Y for (X,Δ'), which is automatically a log terminal model for (X,Δ). Thus we can assume $K_X+\Delta$ is nef, where the statement follows from the base-point-free theorem.

(step 2) If $h^0(\mathcal{O}_X(\lfloor mk(K_X+\Delta)\rfloor + kA))$ is unbounded function of m (for k sufficient divisible), then $K_X+\Delta\sim_{\mathbb{R}} D\ge 0$.

In fact, given any point $x\in X$, by the standard argument, there exsits a large enough m and an effective $\mathbb{R}$-divisor $H\sim_{\mathbb{R}} m(K_X+\Delta)+A$, whose multiplicity near x is greater than n. Then for $t \in [0, m-\epsilon]$

$$(t+1)(K_X+\Delta) = K_X + \frac{m-t}{m}A + B + t(K_X + \Delta + \frac{1}{m}A) \sim_{\mathbb{R}} K_X + \frac{m-t}{m}A + B + \frac{t}{m}H = K_X + \Delta_t.$$

Denote $A' = \frac{\epsilon}{m}A$, for $0 < \epsilon \ll 1$ the following is true

(1) (X, Δ_0) is *klt*,
(2) $\Delta_t \geq A'$
(3) x is in the log canonical center of $(X, \Delta_{m-\epsilon})$, which implies $\mathrm{LCC}(X, \Delta_{m-\epsilon})$ is not contained in $N_\sigma(K_X + \Delta_t) = N_\sigma(K_X + \Delta)$.

So after passing to a log resolution and pertubating Δ_t, we may assume that for some $t_0 \in [0, m-\epsilon]$, X is smooth, Δ_{t_0} is simple normal crossing, satisfying Δ_{t_0} contains an ample divisor and $\lfloor \Delta_{t_0} \rfloor = S$, where S is irreducible and not contained in $N_\sigma(K_X + \Delta_{t_0})$. $K_X + \Delta \sim_{\mathbb{R}} G \geq 0$ if and only if $K_X + \Delta_{t_0} \sim_{\mathbb{R}} D \geq 0$.

(step 3) Now we have $\Delta = S + A + B$ an *snc* divsor on smooth variety X. $S \not\subset N_\sigma(K_X + \Delta)$, i.e., $\forall \epsilon > 0$, $S \not\subset Bs(K_X + \Delta + \epsilon A)$. Run a MMP with scaling of A,

$$\begin{array}{ccccc} X & \overset{\phi_\epsilon}{\dashrightarrow} & Y_\epsilon & \dashrightarrow & Y_{\epsilon'} \\ & & & & \\ S & \dashrightarrow & T & = & T \end{array}$$

A priori, we do not know this MMP with scaling will terminate on X, however, it can be seen that it follows by our assumptions $(0.1)_{(n-1)}$ and 0.2_n that we know it terminates on S. Furthermore, $S \not\subset Bs(K_X + \Delta + \epsilon A)$ which implies that S is not contracted. Denote the divisor $\phi_{\epsilon *}(S)$ by T and $\phi_{\epsilon *}(\Delta)$ by Δ_ϵ, $(K_{Y_\epsilon} + \Delta_\epsilon)|_T = K_T + \Theta$. Then $K_T + \Theta$ is nef, hence $K_T + \Theta \sim_{\mathbb{R}} G \geq 0$ by the base-point-free theorem. The remaining work is lifting G to Y_ϵ then to X.

If $K_X + \Delta \in \mathrm{Div}_{\mathbb{Q}}(X)$, we may assume $m(K_{Y_\epsilon} + \Delta_\epsilon)$ is integral and Cartier on a neighborhood of T and $h^0(T, m(K_T + \Theta)) > 0$. Then by the follow exact sequence,

$$0 \to \mathcal{O}_{Y_\epsilon}(m(K_{Y_\epsilon} + \Delta_\epsilon) - T) \to \mathcal{O}_{Y_\epsilon}(m(K_{Y_\epsilon} + \Delta_\epsilon)) \to \mathcal{O}_T(m(K_T + \Theta)) \to 0,$$

it suffices to prove $H^1(\mathcal{O}_{Y_\epsilon}(m(K_{Y_\epsilon} + \Delta_\epsilon) - T)) = 0$. But $m(K_{Y_\epsilon} + \Delta_\epsilon) - T = K_{Y_\epsilon} + B + (1 - (m-1)\epsilon)A + (m-1)(K_{Y_\epsilon} + \Delta_\epsilon + \epsilon A)$ and we have the pair $(K_{Y_\epsilon}, B + (1-(m-1)\epsilon)A)$ is *klt* and $(m-1)(K_{Y_\epsilon} + \Delta_\epsilon + \epsilon A)$ is big and nef (in the MMP with, we can assume $\epsilon < \frac{1}{m-1}$). In this case, the theorem follows from the Kawamata-Viehweg vanishing theorem.

In the general case, $K_X + \Delta \in Div_{\mathbb{R}}(X)$. We must show that we may write $K_X + \Delta = \sum r_i(K_X + \Delta_i)$ where r_i are real numbers and $\Delta_i \in Div_{\mathbb{Q}}(X)$ so that by a similar argument $K_X + \Delta_i \sim_{\mathbb{R}} D_i \geq 0$. We refer to [BCHM06] for the technical details of the general case.

$\square$

References

[BCHM06] Birkar C.; Cascini P.; Hacon C.; McKernan J.; Existence of minimal models for varieties of log general type, math.AG/0610203. (2006).

Assume Finiteness of M.M $_{n-1}$ & Existence of MM$_n$

Thm Non-Vanishing $\pi : X \to U$ proj (normal quasiproj)

$\dim X = n$ $K_X + \Delta$ klt Δ big over U

~~Assume~~ If $K_X + \Delta$ is pseudo effective / U then

$\exists D \geq 0$ s.t. $K_X + \Delta \sim_{\mathbb{R}, U} D \geq 0$.

If $K_X+\Delta$ nef done BPF, Idea we can run the $K_X+\Delta+tA$ MMP with scaling $t \to 0$.

(ample, ≥ 0) For simplicity assume that $U = \operatorname{Spec} \mathbb{C}$, X smooth Δ snc $\Delta \geq A+B$ (≥ 0, ample)

$\Delta \sim A+B$ $f: Y \to X$ $f^*(K_X+\Delta) + E = K_Y + \Gamma$ $\forall F \geq 0$ - f-exc ample exc

$\Delta' = (1-\epsilon)\Delta + \epsilon A + B$ $f^*A - F = K_Y + (\Gamma - f^*A) + f^*A - \delta F + \delta F \sim K_Y + \Gamma - f^*A + \delta F + A'$ (ample)

<u>Step 1</u> If $h^0(\mathcal{O}_X(\lfloor mk(K_X+\Delta)\rfloor + kA))$ is bdd function of m

(for k suff divisible) then $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$

pf <u>Nakayama</u> $K_X + \Delta \equiv N_\sigma(K_X+\Delta) \geq 0$

($\uparrow$ ~~lim~~ stable fixed divisor $\epsilon \to 0$ of $K_X + \Delta + \epsilon A$)

Set $A' \sim_{\mathbb{R}} A + N_\sigma(K_X+\Delta) - K_X + \Delta$ ample

$\Delta' = \Delta - A + A' \geq 0$

$K_X + \Delta \equiv K_X + \Delta' \sim_{\mathbb{R}} N_\sigma(K_X+\Delta)$ (klt)

So $\exists \phi : X \dashrightarrow Y$ LTM (X, Δ') hence also a LTM (X, Δ)

So we may assume that $K_X + \Delta$ is nef & by BPF

$K_X + \Delta \sim D \geq 0$

<u>Step 2</u> $h^0(\mathcal{O}_X(\lfloor mk(K_X+\Delta)\rfloor + kA))$ is unbounded fn of m

then $\exists$ $0 \leq H \sim_{\mathbb{R}} m(K_X+\Delta) + A$ $\operatorname{mult}_x H > n$

$t \in [0, m-\epsilon]$ $(t+1)(K_X+\Delta) = K_X + \frac{m-t}{m} A + B + t(K_X + \Delta + \frac{1}{m} A)$

$$\sim_{\mathbb{R}} K_X + \underbrace{\frac{m-t}{t} A}_{\geq \frac{\epsilon}{m} A = A'} + B + \frac{t}{m} H = K_X + \Delta_t$$

$K_X + \Delta_0$ klt $\Delta_t \geq A'$ ~~K...~~ $x \in$ Nklt $(K_X + \Delta_{m-\epsilon})$

So after passing to a log resolution, may assume

that ~~$K_X+\Delta_t$~~ for some $t_0 \in [0, m-\epsilon]$. X is smooth Δ_{t_0} snc.

$\Delta_{t_0} \geq$ Ample $\lfloor \Delta_t \rfloor = S$ irred div $\not\subset N_\sigma(K_X+\Delta_t)$ $K_X + \Delta_t \sim_{\mathbb{R}} G \geq 0$ iff $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$

Step 3 Now we have $\Delta = S + A + B$. ~~Assume~~ since X smooth
$S \not\subset N_\sigma(K_X+\Delta)$ ie $\forall \epsilon > 0$ $S \not\subset B_s(K_X+\Delta+\epsilon A)$
assume $(K_X+\Delta) \in \mathrm{Div}_\mathbb{Q}(X)$
~~$\forall \epsilon > 0$ we have a MMP $\phi : X \dashrightarrow Y_\epsilon$ of $(K_X+\Delta+\epsilon A)$~~
Run a MMP with scaling of A
this terminates $\forall \epsilon > 0$, & on S (S not contracted $\forall \epsilon > 0$)

$$X \overset{\phi_\epsilon}{\dashrightarrow} Y_\epsilon \dashrightarrow Y_{\epsilon'}$$
$$S \dashrightarrow T \dashrightarrow T$$

so $\Gamma_\epsilon + A_\epsilon = \phi_{\epsilon *}(\Gamma + A)$ then $(K_{Y_\epsilon} + \Gamma_\epsilon)|_T = K_T + \Theta$

$K_T + \Theta + \epsilon A|_T$ nef $\forall \epsilon > 0$ so $K_T + \Theta$ is nef
hence $K_T + \Theta \sim_\mathbb{R} G \geq 0$ by BPF
We must lift G to Y_ϵ & then to X
If $K_X + \Delta \in \mathrm{Div}_\mathbb{Q}(X)$ $h^0(\mathcal{O}_T(m(K_T+\Theta))) > 0$ integral &
& we may assume that $m(K_{Y_\epsilon}+\Delta_\epsilon)$ is Cartier near S
&

$$0 \to \mathcal{O}_X(m(K_{Y_\epsilon}+\Delta_\epsilon) - T) \to \mathcal{O}_X(m(K_{Y_\epsilon}+\Delta_\epsilon)) \to \mathcal{O}_S(m(K_T+\Theta)) \to$$

Need $H^1(\text{———}) = 0$

$$m(K_Y+\Delta) - T = K_Y + A + B + (m-1)(K_Y+\Delta)$$
$$= \underbrace{K_Y + B}_{\text{klt}} + \underbrace{(m-1)\left(K_Y + \Delta + \frac{A}{m-1}\right)}_{\text{nef + big.}}$$

Otherwise if $K_X + \Delta \notin \mathrm{Div}_\mathbb{Q}(X)$ write $K_X + \Delta = \sum r_i (K_X + \Delta_i)$, $r_i \in \mathbb{R}$

& work much harder

$r_i(K_X + \Delta_i)$ integral

$\Delta' = S + A + B' + tH$ $\|B' - B\| < \alpha t$

$m_i(K_T + \Theta_i)$ bpf

$\phi : V \to W$

$\Delta' \dashrightarrow \Theta$

~~$D = N + R = \sum a_i(t_i) = R_0 + R_E$ $R' = \sum r_i(t_i)$~~

~~$R_m = \sum m_i(t_i)$ $E = nR - R_m = \sum e_i(t_i)$ $0 < e_i < 1$~~

$\mathrm{mult}_C E = \mathrm{mult}_C R$

Existence of MM n $K_X + D$ klt $\overset{D \text{ big}}{\vee}$ $K_X + D \sim_{\mathbb{R}} D \geq 0$
then (X, D) has a log terminal model

Finiteness$_{n-1}$ $K_X + A + \sum t_i D_i$ X smooth $A + \sum D_i$ SNC
$\{Y \mid Y$ is a $\underset{\text{WLC}}{\underline{\text{LTM}}}$ $(X, A + \sum t_i D_i)$ $t \in [0,1]^k\}$ is finite