On minimal log discrepancies

(X, D) $\begin{cases} X \text{ normal}, \ \dim X =: d \\ D \geq 0. \ K_X + D \ \mathbb{R}\text{-Cartier} \end{cases}$

v_E alg. val'n $\quad \begin{array}{ccc} \bar{X} & \supset & E \\ f \downarrow & & \downarrow \\ X & \supset & c_X(E) \end{array}$ log resol'n $\quad K_{\bar{X}} + \bar{D} = f^*(K_X + D) + \underset{f\text{-excep.}}{A}$

centre $\qquad$ log disc $a_E(X, D) := 1 + \mathrm{mult}_E(A - \bar{D})$

$X \supset Z \quad \mathrm{mld}_Z(X, D) := \min\limits_{c_X(E) \subset Z} \{a_E(X, D)\} \in \mathbb{R}_{\geq 0} \cup \{-\infty\}$

$\begin{cases} \mathrm{mld} \geq 0 & \Leftrightarrow (X, D) \text{ lc along } Z \\ \quad > 0 & \Leftarrow \qquad \text{klt} \end{cases}$

mld's measure sing's :

large mld $\leftrightarrow$ mild sing.

$X \ni x$ surface $\quad \mathrm{mld}_x X = 2 \quad \Leftrightarrow \quad X \ni x$ sm.

$\qquad\qquad\qquad\qquad \geq 1 \qquad\qquad$ Du Val

$\qquad\qquad\qquad\qquad > 0 \qquad\qquad$ quoti sing.

term'n of flips $\Leftarrow$ 2 conj's on mld's

Shokurov $\begin{cases} \text{(LSC)} \ X \ni x \mapsto \mathrm{mld}_x(X, D) \quad \text{lower s.c.} \\ \text{(ACC)} \ \{\mathrm{mld}_{Z'}(X', D') \mid (X', D') \text{ isom. in codim 1 to } \overset{\text{fix'd}}{(X, D)}\} \text{ acc.} \end{cases}$

$\uparrow$

$\cdots \to X_i \overset{\phi_i}{\dashrightarrow} X_{i+1} \dashrightarrow \cdots \qquad b_i := \inf\limits_{j \geq i} \{\mathrm{mld}_{E_j}(X_j, D_j)\} = \mathrm{mld}_{\ni Z}(X_i, D_i)$

$\ \ \cup$

$E_i := \mathrm{Exc}(\phi_i)$

$b_i \leq b_{i+1} \leq \cdots \xrightarrow{\text{(ACC)}} \exists\, b$

Choose m, so. $\begin{cases} \mathrm{mld}_\xi(X_i, D_i) = b, \ \xi \in E_i \Rightarrow \dim \overline{\{\xi\}} \geq m \\ \dim \overline{\{\xi\}} = m, \text{ w inf. many } i. \end{cases}$

$W_i := \overline{\{\xi \mid \mathrm{mld}_\xi(X_i, D_i) = b\}}$

$\uparrow$ can relax to $\leq$ by (LSC).

$W_i \dashrightarrow W_{i+1} \dashrightarrow$ isom. in. codim $\dim W_i - m$.

BCHM. MMP for (X, D) D big.

not known : $\exists$ min. model for $\kappa < d$.

Even in dim 3, $\exists$ min. model $\Leftarrow$ term'n of flip's $\Leftarrow$ analysis on mld's.

class'n of term. sing's.

ACC in dim 2, by Alexeev

(LSC) $\to$ mld $\leq d$ } boundedness of mld's
(ACC) $\to$ mld bounded. } fundamental problem.

may assume $D=0$, X Gor. lc. (not klt.)
By BCHM, $\exists_{\bar{x}\in} (\bar{X}, \bar{D})$ lc $\mathbb{Q}$-fact
$f \downarrow$ $K_{\bar{X}}+\bar{D}=f^*(K_X+D)$
$x \in (X, D)$ lc.

index 1 cover.

$\mathrm{mld}_x(X,D) \leq \mathrm{mld}_{\bar{x}}(\bar{X},\bar{D}) \leq \mathrm{mld}_{\bar{x}} \bar{X} \leq \mathrm{mld}_{\bar{x}'} \bar{X}'$

mld's have description in terms of motivic int'n.

$(\wedge \Omega_X) \longrightarrow J_X \omega_X$.

(Sch) $\to$ (Sch) has right adj. J_n.

$X \mapsto X \times \mathrm{Spec}\, k[t]/(t^{n+1})$

$J_\infty X := \varprojlim J_n X$ $\leadsto$ motivic int'n. $\mu_X : B_X \longrightarrow \widehat{M}_{\mathbb{Q}}$ $K_0(\mathrm{Sch}/k)$ $\mathbb{L} := [\mathbb{A}^1]$

measurable set on $J_\infty X$ } local'd by $\{\mathbb{L}^n\}$ M

$$\pi^X \left(\begin{array}{c} J_\infty X \\ \downarrow \\ J_n X \\ \downarrow \pi_n^X \\ J_0 X = X \end{array} \right.$$

X lc $\Leftrightarrow$ $\mathrm{Int}(\varepsilon) := \int_{\pi^{-1}(Z)} \mathbb{L}^{(1-\varepsilon)J_X} d\mu_X$ $\quad M_{\mathbb{Q}} := \bigoplus_{\gamma \in \mathbb{Q}\cap[0,1)} M \mathbb{L}^\gamma$

has a limit.

$\mathrm{mld} := \lim_{\varepsilon \to +0} -\dim \cancel{\int_{\pi^{-1}(Z)} \mathbb{L}^{1-\varepsilon}} \mathrm{Int}(\varepsilon)$.

[EMY]
LSC for lci.
inv. adj

large mld $\leftrightarrow$ small $\dim (\pi_n^X)^{-1}(Z)$

~~Evidence~~ $\leftrightarrow$ small $\dim (\pi_1^X)^{-1}(x) = \mathrm{emb}\dim_x X$

$\exists$ lci: $J_X = J_X'$ Jacobian

in gen'l. $X \subset Y$ lci $= X \cup C$. $\subset A$

$J_Y'|_X = C_{X/Y} \cdot J_X$ $\quad I_Y + (c) = I_C$, c ~~generated by~~ $C_{X/Y}$

$J_X' = (\sum C_{X/Y}) \cdot J_X$

weak lci defect ideal, co-supp'd on non-lci locus.

ord c const. around γ

$\gamma \in J_\infty X$
$\downarrow$
$\gamma_n \in J_n X \subset J_n Y$

difference of c $\leadsto$ likely $\dim_{\gamma_n} \pi_Y^{-1}(x) \geq \mathrm{nd} + \mathrm{ord}_\gamma C$

$\dim_{\gamma_n} \pi_X^{-1}(x) \geq nd$

$\Downarrow$ the formula
$\mathrm{mld}_x X \leq d$

A free resol'n.
$\downarrow c$
A

)

d
$\{X, D\}\}$ acc.

j's.
$j$lexeev

On minimal log discrepancies

(X, D) $\begin{cases} X \text{ normal}, & \dim X =: d \\ D \geq 0, & K_X + D \ \mathbb{R}\text{-Cartier} \end{cases}$

v_E alg. val'n. $\bar{X} \supset E$ log resol'n. $K_{\bar{X}} + \bar{D} = f^*(K_X + D) + \underset{f\text{-excep}}{A}$

$f \downarrow \quad \downarrow$

$X \supset C_X(E)$ centre   log disc $a_E(X, D) := 1 + \mathrm{mult}_E(A - \bar{D})$

$X \supset Z \quad \mathrm{mld}_Z(X, D) := \min_{C_X(E) \subset Z} \{a_E(X, D)\} \in \mathbb{R}_{\geq 0} \cup \{-\infty\}$

$\begin{cases} \mathrm{mld} \geq 0 \Leftrightarrow \text{lc along } Z \\ \qquad > 0 \Leftarrow \text{klt.} \end{cases}$

- mld's measures sing's : large mld $\leftrightarrow$ mild sing.
  $X \ni x$ surface. $\mathrm{mld}_x X = 2 \Leftrightarrow$ sm.
  ≥ 1 DuVal
  > 0 quoti.
- term'n of flips $\overset{\text{Shokurov}}{\Leftarrow}$ 2 conj's on mld's $\begin{cases} \text{(LSC)} & X \ni x \mapsto \mathrm{mld}_x(X, D) \\ \text{(ACC)} & \{\mathrm{mld}_x(X, D = \sum d_i D_i) \mid \{d_i\} \in \mathcal{D}\} \end{cases}$

$\cdots \dashrightarrow X_i \overset{\phi_i}{\dashrightarrow} X_{i+1} \dashrightarrow \cdots \quad b_i := \inf_{j \geq i} \{\mathrm{mld}_{E_j}(X_j, D_j)\} = \mathrm{mld}^{\exists}_{Z}(X_i, D_i)$

$\mathrm{Exc}(\phi_i) =: E_i$

$b_i \leq b_{i+1} \leq \cdots \overset{\text{(ACC)}}{\longrightarrow} = b \qquad W_i^b := \{\mathrm{mld}_\xi(X_i, D_i) = b, \ d_i - \xi = m\}$ ($\leq$ relax by (LSC))

$m := \min_i \{\max\{\dim \overline{\{\xi\}} \mid \xi \in E_j, \ \mathrm{mld}_\xi(X_j, D_j) = b, \ j \geq i\}\}$

$W_i := \{\xi \mid \mathrm{mld}_\xi(X_i, D_i) = b, \ \dim \overline{\{\xi\}} = m\}$ $W_i^b \dashrightarrow W_{i+1}^b$ cycle argument

(= can relax to $\leq$ by (LSC))

$W_i \dashrightarrow W_{i+1}$ isom in $\dim > m$, contra. in $\dim = m$.

BCHM : MMP w/ boundary big.
 not known $\exists$ min. models for $K < d$.
Even in dim 3, $\exists$ min. models $\Leftarrow$ term'n of flips
 $\Leftarrow$ ~~difficulty argument~~
 $\Leftarrow$ analysis of mld's $\begin{cases} \text{class'n of term. sing's.} \\ \text{ACC in codim 2 (Alexeev)} \end{cases}$
 "difficulty"

$\left.\begin{array}{l} \text{(LSC)} \Rightarrow \mathrm{mld} \leq d \\ \text{(ACC)} \Rightarrow \mathrm{mld} \text{ bounded} \end{array}\right\}$ boundedness of mld : basic problem.
 may assume $D = 0$, X lc, Gor.
 ($D = 0$: lc $\mathbb{Q}$-fact'n by BCHM; lc: trivial; Gor: index-1-cover)

. can't assume X klt (i.e. cano.)

A − $\bar{D}$)

)

$\{d_i\} \in \mathcal{D}$ }

$i) = b.\ d_i - \xi = m$ }
$\leq$
relax by (LSC)

V^b_{i+1} cycle argument.

sing's.
(Alexeev)

over

X klt (i.e. cano.)

mld : description in terms of motivic integ'n

$F_n : (Sch/k) \to (Sch/k)$ right adj J_n

$X \mapsto X \times \operatorname{Spec} k[t]/(t^{n+1})$

$J_\infty X := \varprojlim J_n X$ $\qquad \mu : \mathcal{B} \longrightarrow \widehat{M}_{\mathbb{Q}}$:= the completion of $M_{\mathbb{Q}} := \bigoplus_{\gamma \in \mathbb{Q} \cap [0,1)} M \cdot \mathbb{L}^\gamma$,

π^X $\downarrow$

$J_n X \underset{\text{Hom}_{sch}(\operatorname{Spec},\,X)}{\cong} \{\operatorname{Spec} k[t]/(t^{n+1}) \to X\}$ $\qquad$ (M := the local'n of $K_0(Sch/k)$ by $\{\mathbb{L}^n\}_{n \geq 0}$, $\mathbb{L} := [\mathbb{A}^1]$)

$\downarrow \pi^X_n$

$J_0 X = X$ $\quad X \ni x$ lc $\Leftrightarrow$ $\operatorname{Int}(\varepsilon) := \int_{(\pi^X)^{-1}(x)} \mathbb{L}^{(1-\varepsilon)\operatorname{ord} J_X} d\mu_X$ has a limit, $\forall \varepsilon > 0$.

[EMY] $\operatorname{mld}_x X = -\lim_{\varepsilon \to +0} \dim \operatorname{Int}(\varepsilon)$

means : large mld $\leftrightarrow$ small $J_n X$

$\bigwedge^d \Omega_X \twoheadrightarrow J_X \omega_X$. $J_X = J'_X$ Jacobian if X lci $\overset{\text{EMY}}{(\Rightarrow}$ (LSC), inv. adj.)

in general, $X \subset Y$ l.c.i. of dim d
$\quad\quad X \overset{\shortparallel}{\cup} C$

$J'_Y|_X = C_{X/Y} \cdot J_X$ by Grothendieck duality.

$J'_X (= \Sigma J'_Y|_X) = \underbrace{(\Sigma C_{X/Y})}_{\text{lci defect ideal}} \cdot J_X$ (co-supported on the non-lci locus of X)

X Gor $\Rightarrow$ $C_{X/Y}$ principal, say generator c

$$\begin{array}{cccccl}
\text{ord}_c \text{ const. near } \gamma \ \ \gamma & \in & J_\infty X & \subset & J_\infty Y & \\
\downarrow & & \downarrow & & \downarrow & \\
\gamma_n & \in & J_n X & \subset & J_n Y & \text{difference "c"} \\
\downarrow & & \downarrow & & & \\
x & \in & X & & & \dim_{\gamma_n} (\pi^X_n)^{-1}(x) \geq nd
\end{array}$$

$\mapsto$ expect $\dim_{\gamma_n} (\pi^Y_n)^{-1}(x) \geq nd + \operatorname{ord}_\gamma c$
$\Downarrow$ by EMY
$\operatorname{mld}_x X \leq d$

(S. Takagi's idea)

- free resol'ns $0 \to \mathcal{O}_A \to \quad \to \mathcal{O}_A \to I_Y \to 0$

$X \subset Y \subset A$ $\quad \downarrow \times c \quad \cdots \quad \| \quad \cap$

$0 \to \mathcal{O}_A \to \quad \to \mathcal{O}_A \to I_X \to 0$

R-R technique ($\Leftarrow$ multiplier ideal, ~~$\Leftrightarrow$ lc threshold~~) $\leftrightarrow$ gen'd test ideal $\quad$ lc thd $\leadsto$ only lc thd find mld in pos. char.

large mld $\leftrightarrow$ small $J_n X$

$\leftrightarrow$ small $J_1 X$ ($\Leftrightarrow$ small $\operatorname{embdim}_x X$)

$\operatorname{embdim}_x X$ bounded $\underset{\text{R-R}}{\Rightarrow}$ $\operatorname{mult}_x X$ $(:= (H_1, \cdots, H_d)_x)$ bounded $\underset{\text{R-R}}{\Rightarrow}$ mld bounded

On minimal log discrepancies

(X, D) $\begin{cases} X \text{ normal}, \dim = d \\ D \geq 0.\ K_X + D\ \mathbb{R}\text{-Cartier}. \end{cases}$

$$\begin{array}{ccc} \bar{X} & \supset & E \\ f\downarrow & & \downarrow \\ X & \supset & C_X(E) \end{array}$$

log resol'n $K_{\bar{X}} + \bar{D} = f^*(K_X + D) + A$ (A f-excep.)

$C_X(E)$ centre. log disc $a_E(X, D) := 1 + \mathrm{mult}_E(A - \bar{D})$

$X \supset Z$. $\mathrm{mld}_Z(X, D) := \min_{C_X(E) \subset Z} \{a_E(X, D)\} \quad \in \mathbb{R}_{\geq 0} \cup \{-\infty\}$

$\begin{cases} \mathrm{mld} \geq 0 \Leftrightarrow \text{lc along } Z \\ \phantom{\mathrm{mld}} > 0 \Rightarrow \text{klt} \end{cases}$

- mld's measure sing's : large mld $\leftrightarrow$ mild sing.
  $X \ni x$ surface. mld $= 2$, ≥ 1, > 0
  sm DuVal quoti.
- term'n of flips $\overset{\text{Shokurov}}{\Leftarrow}$ conj's on mld's $\begin{cases} (\text{LSC}) \quad X \ni x \mapsto \mathrm{mld}_x(X, D) \\ (\text{ACC}) \quad \{\mathrm{mld}_x(X, D = \sum d_i D_i) \mid \{d_i\} \in \mathcal{D}\} \end{cases}$
  $\uparrow$

$X_i \overset{\phi_i}{\dashrightarrow} X_{i+1} \dashrightarrow$, $b_i := \inf_{j \geq i} \{\mathrm{mld}_{E_j}(X_j, D_j)\} = \mathrm{mld}_{\ni Z}(X_i, D_i)$
$\cup$
$E_i := \mathrm{Exc}(\phi_i)$

$b_i \leq b_{i+1} \leq \cdots \xrightarrow{(\text{ACC})} = b.$

$W_i^m := \{\xi \mid \dim \overline{\{\xi\}} = m,\ \mathrm{mld}_\xi(X_i, D_i) = b\}$
can relax to $\leq$. by (LSC)

apply cycle argument to $W_i^m \dashrightarrow W_{i+1}^m \dashrightarrow$

BCHM : MMP w/ boundary big.
not known $\exists$ min. models of $K < d$.
Even in dim 3, $\exists$ min. models $\Leftarrow$ term'n of flips $\Leftarrow$ "difficulty" analysis on mld's. $\begin{cases} \text{classi'n of term. sing's.} \\ \text{ACC in dim 2 (Alexeev} \end{cases}$

(LSC) $\Rightarrow$ mld $\leq d$
(ACC) $\Rightarrow$ mld bounded

basic problem : bddness of mld's.
may assume $D = 0$, X Gor, lc
lc $\mathbb{Q}$-fact'n by BCHM index-1-cov. triv.

mld : descript'n in terms of motivic integ'n

$J_\infty X = \varprojlim J_n X$
$\downarrow$
$J_n X \underset{\text{set-th}}{=} \mathrm{Hom}(\mathrm{Spec}\, k[t]/(t^{n+1}), X)$
$\downarrow \pi_n^X$
X

($\pi^X : J_\infty X \to X$)

$\mu : \mathcal{B} \longrightarrow \widehat{M}_{\mathbb{Q}}$, $M_{\mathbb{Q}} = \bigoplus_{\gamma \in [0,1)} M[\mathbb{L}^\gamma]$.

M = the local'n of $K_0(\mathrm{Sch}/k)$ w.r.t. $\{\mathbb{L}^n\}_{n \geq 0}$, $\mathbb{L} := [\mathbb{A}^1]$.

[EMY] $X \ni x$ lc $\Leftrightarrow$ $\mathrm{Int}(\varepsilon) := \int_{(\pi^X_\infty)^{-1}(x)} \mathbb{L}^{(1-\varepsilon)\mathrm{ord}_{J_X}} d\mu_X$ has a limit,

$$\mathrm{mld}_x X = -\lim_{\varepsilon\to+0} \dim \mathrm{Int}(\varepsilon)$$

$\rightsquigarrow$ large mld $\leftrightarrow$ small ($J_n X$ + ord_{J_X})
$\rightarrow$ small $J_1 X$ (i.e. $\mathrm{embdim}_x X$)

$\wedge^d \Omega_X \twoheadrightarrow J_X \omega_X$. $J_X = J_X'$ Jacobian if X lci. ($\overset{\mathrm{EMY}}{\Rightarrow}$ (LSC), inv. adj.)

in gen'l. $X \subset Y$ lci, dim d

$Y = X \cup C$

$J_Y'|_X = C_{X/Y} \cdot J_X$ (by Grothendieck duality)
($C_{X/Y}$: conductor)

$J_X' = \sum_Y J_Y'|_X \overset{?}{=} \underbrace{(\sum_Y C_{X/Y})}_{\text{lci defect ideal (co-supp'd on the non-lci locus of } X)} \cdot J_X$

X Gor $\Rightarrow$ $C_{X/Y}$ principal, say generator c

ord_c const. around γ $\quad \gamma \in J_\infty X$

$\gamma_n \in J_n X \subset J_n Y$ (difference "by c")

$x \in X$ $\quad \dim_{\gamma_n} (\pi_n^X)^{-1}(x) \geq nd$.

$\longrightarrow$ can expect : $\dim_{\gamma_n} (\pi_n^Y)^{-1}(x) \geq nd + \mathrm{ord}_\gamma c$
$\overset{?}{\underset{\mathrm{EMY}}{\Rightarrow}}$ $\mathrm{mld}_x X \leq d$

· R-R technique

$\left.\begin{array}{l}\mathrm{embdim}_x X \text{ bdd.} \\ X \text{ Gor lc}\end{array}\right\} \overset{\text{R-R}}{\Rightarrow} \mathrm{mult}_x X$ $(:= (H_1 \cdots H_d)_x)$ $\overset{\text{R-R}}{\Rightarrow}$ ~~$\mathrm{mld}_x X$ bdd~~
bdd

$-F|_E$ big.
$K_{\bar X} + E - \ell F \to K_E - \ell F|_E$
bdd $\quad \frac{a_E(X)}{b(m_E(X))}$ bdd

$\bar X \quad f^*H = \bar H + F$. $\quad 0 \to \mathcal{O}_{\bar X}(K_{\bar X} - (\ell+1)F) \to \mathcal{O}_{\bar X}(K_{\bar X} - \ell F) \to \mathcal{O}_F(K_{\bar X}|_F - \ell F|_F) \to 0$

$f\downarrow \quad \bar H$ f-free $\quad$ KV vanishing $R^i f_* \mathcal{O}_{\bar X}(K_{\bar X} - \ell F) = 0 \quad (\ell \geq 1)$

$X \supset H \ni x$ $\quad$ X lc $\Rightarrow$ $f_* \mathcal{O}_{\bar X}(K_{\bar X} - \ell F) \supset m_x^{\ell+1} \cdot \mathcal{O}(K_X)$.

$\Rightarrow \chi(F, K_{\bar X}|_F - \ell F|_F) = \dim f_* \mathcal{O}_{\bar X}(K_{\bar X} - \ell F) / f_* \mathcal{O}_{\bar X}(K_{\bar X} - (\ell+1)F)$
$\ell \geq 1$ $\quad$ finite possibilities for bdd embdim.

$\Rightarrow \mathrm{mult}_x X = (-1)^{d-1}(F^d)$ bdd

· R-R suited to multiplier ideals $\leftrightarrow$ lc thd's (not mld's)

$\Updownarrow$ Hara. Yoshida. Takagi.

gen'd test ideals (in char p) . . not use $K + D$ $\mathbb{R}$-Cartier

what in char p ?

$\{d_i\} \in \mathcal{D}\}$

ss'n of term. sing's
C in dim 2 (Alexeev)

8].

of $K_0(Sch/k)$

$n \geq 0$, $\mathbb{L} := [\mathbb{A}^1]$

On minimal log discrepancies

$(X, D) \begin{cases} X \text{ normal}, \ \dim = d \\ D \geq 0. \ K_X + D \ \mathbb{R}\text{-Cartier} \end{cases}$

$\bar{X} \supset E$ log resol'n $K_{\bar{X}} + \bar{D} = f^*(K_X + D) + A$ f-excep

$f \downarrow \quad \downarrow$

$X \supset C_X(E)$ centre log disc $a_E(X, D) := 1 + \mathrm{mult}_E(A - \bar{D})$

$X \supset Z$ $\mathrm{mld}_Z(X, D) := \min_{C_X(E) \subset Z} \{a_E(X, D)\} \in \mathbb{R}_{\geq 0} \cup \{-\infty\}$

$\begin{cases} \mathrm{mld} \geq 0 \Leftrightarrow \text{lc along } Z \\ \quad > 0 \Rightarrow \text{klt} \end{cases}$

- mld's measure sing's : large mld $\leftrightarrow$ mild sing.
  $X \ni x$ surface mld $= 2, \geq 1, > 0$
  sm. Du Val quoti.
- term'n of flips $\overset{\text{Shokurov}}{\Leftarrow}$ conj's on mld's $\begin{cases} \text{(LSC)} & X \ni x \mapsto \mathrm{mld}_x(X, D) \\ \text{(ACC)} & \{\mathrm{mld}_x(X, D = \Sigma d_i D_i) \mid (d_i) \in \mathcal{D}\}. \end{cases}$

global geom $\uparrow$ local

$\left\{ \begin{array}{l} X_i \overset{\Phi_i}{\dashrightarrow} X_{i+1} \dashrightarrow . \ b_i := \inf_{j \geq i} \{\mathrm{mld}_{E_j}(X_j, D_j)\} = \mathrm{mld}_{\ni Z}(X_i, D_i) \\ E_i := \mathrm{Exc}(\Phi_i) \\ \qquad b_i \leq b_{i+1} \leq \cdots \overset{\text{(ACC)}}{\longrightarrow} = b \\ W_i^m := \{\xi \mid \dim \overline{\{\xi\}} = m, \ \mathrm{mld}_\xi(X_i, D_i) = b\} \\ \text{can apply cycle arg. to } W_i^m \dashrightarrow W_{i+1}^m \dashrightarrow \text{ can relax to } \geq \text{ by (LSC)} \end{array} \right.$

BCHM : MMP w/ boundary big.

not known $\exists$ min. models of $K < d$

Even in dim 3, $\exists$ min. models $\Leftarrow$ term'n of flips $\Leftarrow$ analysis on mld's $\begin{cases} \text{class'n of term. sing} \\ \text{OR} \\ \sim\sim\text{ACC in dim 2 (Alexe}\sim\sim \end{cases}$

$\left. \begin{array}{l} \text{(LSC)} \Rightarrow \mathrm{mld} \leq d \\ \text{(ACC)} \Rightarrow \mathrm{mld} \text{ bounded} \end{array} \right\}$ basic problem : bddness of mld's.
may assume $D = 0$, X Gor. lc
lc $\mathbb{Q}$-fact'n by BCHM index 1 cov triv.

mld's : description in terms of motivic integ'n.

$J_\infty X = \varprojlim J_n X.$

$\pi^X \quad J_n X \underset{\text{set th}}{=} \mathrm{Hom}(\mathrm{Spec}\, k[t]/(t^{n+1}), X)$

$\pi_n^X \downarrow \quad X$

$\mu : \mathcal{B} \longrightarrow \hat{M}_{\mathbb{Q}}$ the compl'n of $M_{\mathbb{Q}} \cdot \bigoplus_{\gamma \in \mathbb{Q} \cap [0,1)} M \mathbb{L}^\gamma$.

$M :=$ the local'n of $K_0(\mathrm{Sch}/k)$ w.r.t. $\{\mathbb{L}^n\}_{n \geq 0}$, $\mathbb{L} = [\mathbb{A}^1]$

$\overset{d}{\wedge} \Omega_X \twoheadrightarrow J_X \omega_X$

EMY : $X \ni x$ lc $\Leftrightarrow \mathrm{Int}(\varepsilon) := \int_{(\pi^X)^{-1}(x)} \mathbb{L}^{(1-\varepsilon)\mathrm{ord}_{J_X}} d\mu_X$ has a limit $\forall \varepsilon > 0$.

$\mathrm{mld}_x X = -\lim_{\varepsilon \to +0} \dim \mathrm{Int}(\varepsilon)$, inv. def'd by $\hat{\mathcal{O}}_{x,X}$

J_X In ord R-R R-R

$\rightsquigarrow$ large mld $\leftrightarrow$ small $[J_n X + \mathrm{ord}_{J_X}]$
$\to$ small $J_1 X$ (i.e. $\mathrm{embdim}_x X$)

· $J_X = J_X'$ Jacobian if X lci $\overset{\text{EMY}}{\Rightarrow}$ (LSC) inv. adj.
In gen'l, $X \subset Y$ lci of dim d
$X \overset{\shortparallel}{\cup} C$ $\quad J_Y'|_X = C_{X/Y} \cdot J_X$ by Grothendieck duality.
conduct.

$J_X' \ (= \sum_Y J_Y'|_X) = \underbrace{(\Sigma C_{X/Y})}_{} \cdot J_X$
lci defect ideal (co-supp'd on the non-lci locus on)

$\mathrm{ord}_{C_{X/Y}}$ const. around γ $\quad \gamma \in J_\infty X$, $\gamma_n \in J_n X \subset J_n Y$, $x \in X$
difference "by $C_{X/Y}$" $\Rightarrow$ expects: $\dim_{\gamma_n}(\pi^Y)^{-1}(x) \geq nd + \mathrm{ord}_{C_{X/Y}}$
$\dim_{\gamma_n}(\pi^X)^{-1}(x) \geq nd.$ $\quad \overset{\Longrightarrow}{\text{EMY}}$ $\mathrm{mld}_x X \leq d$
$C_{X/Y}$ principal (S. Takagi told: can compute in terms of free resol'n)

· R-R technique (X Gor. lc) $\quad$ (Can bd $\mathrm{embdim}_x X$ of mld large?)
bdd $\mathrm{embdim}_x X \overset{\text{R-R}}{\underset{①}{\Rightarrow}}$ bdd $\mathrm{mult}_x X$ $\cancel{(=(H_1 \cdots H_d)_x)}$ $\overset{\text{R-R}}{\underset{②}{\Rightarrow}}$ bdd $\mathrm{mld}_x X$

① $\bar{X} \supset f^*H = \bar{H} + F$ $\quad 0 \to \mathcal{O}_{\bar{X}}(K_{\bar{X}} - (l+1)F) \to \mathcal{O}_{\bar{X}}(K_{\bar{X}} - lF) \to \mathcal{O}_F(K_{\bar{X}}|_F - lF|_F)$
$f \downarrow$ $\quad \bar{H}$ f-free.
$X \supset H \ni x$
KV van. $\Rightarrow R^i f_* \mathcal{O}_{\bar{X}}(K_{\bar{X}} - lF) = 0 \quad (l \geq 1)$
X lc $\Rightarrow f_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} - lF) \supset m_x^{l+1} \cdot f_*\mathcal{O}_{\bar{X}}(K_{\bar{X}})$
$\chi(K_{\bar{X}}|_F - lF|_F)_{l \geq 1} = \dim f_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} - lF) / f_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} - (l+1)F)$
finitely many possib.
$\Rightarrow \mathrm{mult}_x X = (-1)^{d-1}(F^d)$ bdd

② $F = \Sigma m_i E_i$. take $\bar{H}|_{E_i}$ big &
consider $0 \to \mathcal{O}_{\bar{X}}(K_{\bar{X}} - lF) \to \mathcal{O}_{\bar{X}}(K_{\bar{X}} + E_i - lF) \to \mathcal{O}_{E_i}(K_{E_i} - lF|_{E_i}) \to 0$
$\dfrac{a_{E_i}(X)}{m_i} \leq d$
bdd

R-R suited to lc thd's (not mld's)
$\Updownarrow$
multiplier ideals ($\leftrightarrow$ gen'd test ideals in char p)
can relax $\mathbb{R}$-Car cond.
question: what in char p corresponds to mld?

$\mathrm{ld}_x(X, D)$
$\Sigma d_i D_i) \mid (d_i) \in \mathcal{D}\}$.
)

SC)

mld's $\left\{ \begin{array}{l} \text{class'n of term. sing} \\ \text{or} \\ \cancel{\text{ACC in dim 2 (Alexeev)}} \end{array} \right.$

or triv.

$M_{\mathbb{Q}} = \bigoplus_{\gamma \in \mathbb{Q} \cap [0,1)} M \mathbb{L}^\gamma$.
of $K_0(\mathrm{Sch}/k)$
≥ 0, $\mathbb{L} = [\mathbb{A}^1]$

limit $\forall \varepsilon > 0$.