Log canonical thresholds of Fano orbifolds (~~ch~~ Ivan Cheltov)

① X is Fano klt, $\dim X = n$. $G \curvearrowright \mathrm{Aut}(X)$. $|G| < \infty$

Def: $\mathrm{lct}(X,G) = \inf\{\mathrm{lct}(X,D) \mid D \geq 0,\ \mathbb{Q}\text{-divisor } D \equiv -K,\ D \text{ is } G\text{-invariant}\}$

if $G=1$. $\mathrm{lct}(X) = \mathrm{lct}(X,G)$

Ex: (Hwang) X is Rational Hom. space. s.t. $\mathrm{Pic}(X) = \mathbb{Z}H'$

$-K_X = rH$, $r \in \mathbb{N}$. Then $\mathrm{lct}(X) = \frac{1}{r}$

Ex: $n=2$ smooth $K^2=5$. Then $\mathrm{Aut}\,X = S_5 \Rightarrow \begin{cases} \mathrm{lct}(X) = \frac{1}{2} \\ \mathrm{lct}(X, \mathbb{Z}_5) = 4/5 \\ 2 = \mathrm{lct}(X, A_5) = \mathrm{lct}(S_5) \end{cases}$

② Thm (Siu, Tian, Nadel, Demailly, Kollár). Suppose that X is orbifold. Then $\exists$ KE metric on X if $\mathrm{lct}(X,G) > \frac{n}{n+1}$

Rem: (D-K) If X is smooth then $\mathrm{lct}(X,G) = \alpha(X,G)$ Tian

③ Ex: (Siu. Nadel, Tian) X = Fermat cubic in $\mathbb{P}^3$. Then $\mathrm{lct}(X, \mathrm{Aut}(X)) > \frac{2}{3}$

Ex (J-K) $X = X_d \subset \mathbb{P}(\theta_0, \theta_3)$ $d = \sum a - 1$. Then $\mathrm{lct}(X) > \frac{2}{3}$, always except 2 families when X has to be general.

Ex (J-K) $X = X_d \subset \mathbb{P}(\theta_0, \dots \theta_4)$, $d = \sum\theta - 1$. quasi-isom. Then $\exists$ 1936 $(\theta_0, \dots \theta_4)$ s.t $\mathrm{lct}(X) > 3/4$.

④ Conj: X Fano. klt. $\exists D \geq 0$ $D \equiv -K$. $\mathbb{Q}$-divisor s.t $\mathrm{lct}(X,G) = \mathrm{lct}(X,D)$

Conj: $\mathrm{lct}(X)$ upper semi-connected.

⑤ X = Fano, term. $\mathbb{Q}$-factorial. $\mathrm{rk}\,\mathrm{Pic}\,X = 1$ ($k \neq \bar{k}$, $G \subset \mathrm{Aut}$) (Mori-Fiber space)

Def: X is B.R / B.S.R if X is unique / MFS + birational $X = \mathrm{Aut}\,X$

Def: X is B.R. of if mobile M on X $\exists \sigma \in \mathrm{Bir}X$. s.t $(X, \lambda\sigma(M))$ is canonical ~~s.t~~ $\lambda \in \mathbb{Q}^+$ s.t $K + \lambda\sigma(M) \equiv 0$

suppose that X & V both B.R what can we say about $X \times V$?

$\mathrm{Bir}\,X \subset \mathrm{Bir}(X_V) \subset \mathrm{Bir}(X \times V)$

$\cup$

$\mathrm{Bir}(V_X) \supset \mathrm{Bir}(X)$

⑥ X, V. U.B.R

Thm: suppose that $lct(X) \geq 1$, $lct(V) \geq 1$ (Puhklikov)

Then 1) $Bir(X \times V) = \langle Bir\, X_V, Bir\, V_X, Aut\, X \times V \rangle$

2) $X \times V$ is non-Rational

3) α, β are only MFS birational to $X \times V$ - up to $\square$

Ex: $X_{n+1} \subset \mathbb{P}^{n+1}$ $n \geq 5$. general $lct = 1$. X is BSR.

Ex: $X \xrightarrow{2:1} \mathbb{P}^n \supset S_{2n}$ general $n \geq 3 \Rightarrow$ ~~Bir~~ $lct = 1$. X is BSR

⑦ $X_d \subset \mathbb{P}(1, a_1, \cdots a_n)$ $d = \sum a_i$ ⊕ terminal quasi-smooth

Then (RS 3 K) $\Rightarrow \exists$ 95 families of X_d.

Ex: $X_4 \subset \mathbb{P}^4$. $X_6 \subset \mathbb{P}(1^4, 3)$. $X_5(1^4, 2)$

Thm (Corti-P-Reid) if X_d is general, then X_d is U.B.R.

$$Bir\, X_d = Bir\, X_d \otimes X(U)$$

Thm: suppose that X_d is general & $-K^3 \leq 1$ Then $lct(X_d) = 1$

Rem: $K^3 > 1 \Rightarrow X_4, X_6, X_5, X_6 \subset \mathbb{P}(1^3, 2^2)$, $X_7 \subset \mathbb{P}(1^3, 2, 3)$

Ex: $X_{20} \subset \mathbb{P}(1, 1, 4, 5, 10)$ m

$$1 \to (\mathbb{Z}_2 * \mathbb{Z}_2)^m \to Bir(\underbrace{X \times \cdots \times X}_{m}) \to S_m \to 1.$$

Ex: $\mathbb{P}^1 \times X \dashrightarrow \mathbb{P}^3$

$X = dP_5$.

$A_5 \in Bir(\mathbb{P}^3) \neq Aut(\mathbb{P}^3)$