Extension of log pluricanonical systems and semipositivity of pluricanonical systems. (H. Tsuji)

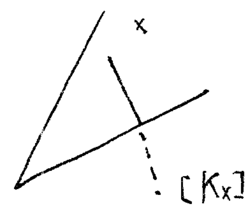
Conj. (Abundance Conj) suppose K_X is pseudo-effective $\Rightarrow \kappa(X) = \nu(X)$

$$\nu(X) = \varlimsup_{m\to\infty} \frac{\log h^0(X, A+mK_X)}{\log m} \quad (A \gg 0)$$

If X is minimal. abundance $\Longleftrightarrow$ K_X is semiample

(1) Find "positive" or "null" direction of $K_X \Rightarrow \exists$ foliation

(2) prove the closeness of leaves.



$$\begin{cases} \dfrac{\partial\omega}{\partial t} = -\mathrm{Ric}_\omega - \omega \\ \omega = \omega_0 \end{cases}$$

Song - Tian

Conjecture (lc): X smooth proj K_X: pseudo-effective. $\nu(X) > 0$

$\Longrightarrow \forall x \in X \ \exists h_x$: singular Herm metric

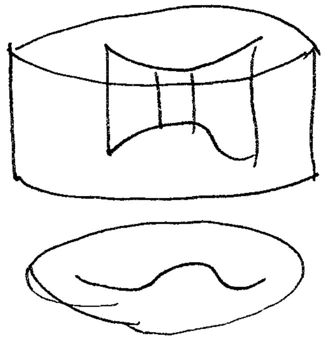
(1) $\Theta_{h_x} \geq 0$ (2) $\nu(\Theta_{h_x}, x) > 0$

Main Thm: (A_j) means Abundance in dim j. if A_m holds $m < n$

$(lc) \Rightarrow A_n$.

Methods: Relate pluricanonical system of LC center to that of ambient space (extension Thm)

	general type	non-general type
extension	Kawamata - Viehweg Vanishing Thm	L^2-extension theorem
required positivity	nef & big	curvature positive
producing loc. can. center	concentration	may be some Monge-Ampère equ.? (conjectural)



$\nu(D) < \nu(X)$

K_D has less direction of positivity than K_X.

Tool: (1) L^2-extension

Theorem (Ohsawa) X smooth proj. $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ effective $\mathbb{Q}$-divisor s.t. $D \in \alpha L$ $(L, h_L) \geq 0$

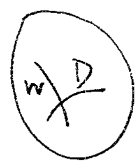
s.t. (X, D) LC but not KLT. W: maximal center of L.C. singularity. We assume W smooth.

$d > \alpha$. $dV_{K+dL|_W}$: C^∞-volume form

$$\Longrightarrow \quad I: A^2(W, dV \otimes h_L^d|_W, dV[\Phi])$$
$$\longrightarrow A^2(X, K_X + dL, dV \otimes h_L^d, dV)$$

what is $d[\Psi]$?

$\Psi = \log h_L^\alpha(\sigma_D \, \sigma_D)$ $\quad D = (\sigma_D)$ mult. hol. section



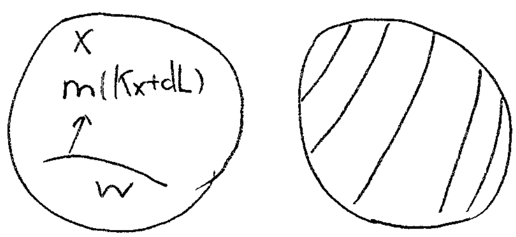
$$\mathrm{Res}(e^{-\Psi} dV)|_W = dV[\Psi]$$

Pluricanonical Version

Thm: $e^{-\varphi} = \dfrac{dV[\Phi]}{dV_W}$ $\quad dV_W$ smooth volume form on W.

$$I: A^2(W, m(K_X + dL)|_W, dV^{-m} \otimes h_L^{dm}|_W e^{-(m-1)\varphi}, dV[\Psi])$$
$$\longrightarrow A^2(X, m(K_X + dL), dV^{-m} \otimes h_L^{dm}, dV)$$

Take $L = K_X$



(2) Subadjunction + Semipositivity

We need to use curvature semipositivity instead of nefness.

canonical sing. hermian metric on log canonical bundle

(X, Δ): lc pair.

$\Longrightarrow \exists \hat{h}_{can}$ on $K_X + \Delta$

(1) $\hat{h}_{can}$ is uniquely determined by (X, Δ)

(2) $\Theta_{\hat{h}_{can}} \geq 0$

(3) $H^0(X, m(K_X+\Delta) \otimes \mathcal{I}(\hat{h}_{can}^m)) \simeq H^0(X, m(K_X+\Delta))$ $\quad \forall$ m is sufficiently divisible $m \geq 1$

AZD
Analytic
Zariski
Decomposition

proj. family

suppose (X,Δ) klt   have horizontal positivity.   VHS or Berndtsson

(H) $\widetilde{\text{hean}} \geq 0$ ~~You~~

$\Longrightarrow$ invariant of plurigenera