KE metrics on Minimal Models

KE metrics on K-pure minimal models.

(Aubin-Yau 1976) M proj. smooth variety / $\mathbb{C}$

$[K_M]$ ample $\Rightarrow$ it contains a ω_{KE} Kähler Einstein metric. $Ricci(\omega_{KE}) = -2\pi\,\omega_{KE}$

$[K_M] \equiv 0 \Rightarrow$ Every Kähler class contains a Ricci flat metric.

Question: Canonical ~~metrics~~ singularities

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§1. Orbifolds $\mathcal{X}$. be a compact complex orbifold with a Kähler metrics (e.g. $\mathcal{X}$ smooth DM stack with projective moduli space.

Aubin-Yau works for such $\mathcal{X}$.

$\mathcal{X}$ be a compact cplx orbifold. $\mathcal{X} \xrightarrow{\pi} X$ a moduli space

$\mathcal{X} = \bigcup_\alpha [U_\alpha / G_\alpha]$ $U_\alpha \subset \mathbb{C}^a$ open subset G_α finite groups acting on U_α faithfully

$X = \bigcup_\alpha U_\alpha / G_\alpha$ $X \supset D$. $D = \sum D_i$

π is ramified ~~with~~ along D. $m_i =$ multiplicity of D_i

$K_{\mathcal{X}} \equiv_{\mathbb{Q}} \pi^*(K_X + \sum_i (1 - \frac{1}{m_i}) D_i)$

$K_{\mathcal{X}}$ ample $\Leftrightarrow K_X + \sum_i (1 - \frac{1}{m_i}) D_i$ ample

Thm: (Chen-Yau, Tian-Yau, Kobayashi) if $K_X + \sum_i (1 - \frac{1}{m_i}) D_i$ ample, ~~with~~ then there exists $T \in |K_X + (1 - \frac{1}{m_i}) D_i|$ smooth outside D, $\forall \alpha: \pi_\alpha: U_\alpha \to X$. $\pi_\alpha^* T_X$ is smooth and defines KE metric on U_i.

T on Δ^n. $D_i = \{z_i = 0\}$ $D = \{z_1 = \cdots = z_p = 0\}$

$$T_z: \sum_{i=1}^{p} \frac{dz_i\, d\bar{z}_i}{|z_i|^{2 - 2/m_i}} + \sum_{p+1}^{n} dz_i\, d\bar{z}_i$$

S: algebraic surface of general type. $\Pi^{min}: S \to S^{min}$

$\Pi^{can}: S \to S^{can}$

S^{can} has isolated crepant singularity. assume $U_a/G_a \subseteq S^{can}$. $\Sigma \xrightarrow{\Pi} S^{can}$ it's smooth

$(\Pi^{-1}(U_\alpha/G_\alpha) = U_\alpha)$

Deligne-Mumford Stack. K_Σ-ample $\Rightarrow$ KE metric on Σ.

The current defined by ω^{Σ}_{KE} on $(S^{can})^{sm}$ extend as a current on S^{can} respecting $K_{S^{an}}$ with hölder continuous potential. On $[U_\alpha/G]^{sm}$

$\omega^{\Sigma}_{KE} = \sqrt{-1}\partial\bar\partial\varphi|_{U_\alpha/G}$ smooth plurisubharmonic. φ hölder continuously on U_α/G.

§2. Weak solution by Monge-Ampère equation

[B-T-1982] $\Omega \subset \mathbb{C}^n$ open set $\varphi \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$

$(i\partial\bar\partial\varphi)^p = i\partial\bar\partial(\varphi(i\partial\bar\partial\varphi)^{p-1})$ current with measure coefficient

$(i\partial\bar\partial\varphi)^n$ is $n \geq 0$ measure $\underset{\text{analytic subset}}{Z \subset \Omega}$ $\int_Z (i\partial\bar\partial\varphi)^n = 0$

This works also for singular spaces (normal)

V normal cplx space

Strong Kähler metric on $V \leftarrow (\underset{\text{open cover}}{U_i}, \varphi_i)$ φ_i smooth strongly plurisubharmonic function

U_i. $\partial\bar\partial\varphi_i = \partial\bar\partial\varphi_j$ on $U_i \cap U_j$. positive current with C^0-potentials.

L holomorphic line bundle, $e^{-\varphi}$ singular metric

$\rightsquigarrow (U_i, \varphi_i)$ $\{i\partial\bar\partial\varphi\} = c_1(L)$

Notation. φ function $T \geq 0$ current on V. $T + i\partial\bar\partial\varphi = (U_i, \varphi_i + \varphi)$

Def: V normal complex space. ω Kähler-class on V.

$\varphi \in C^\infty(V)$ $(\omega + i\partial\bar\partial\varphi)^n = \mu$ μ = measure on V

if $\omega + i\partial\bar\partial\varphi \geq 0$ Monge-Ampère measure is μ.

Thm(EGZ, Z. Zhang) V normal cplx space $[\omega]$ - Kähler class on V. $f \in L^{\varphi}(V, \omega^n)$ $\varphi > 0$ $f \geq 0$

$\exists!\ \varphi \in C^0(V)$ $(\omega + i\partial\bar\partial\varphi)^n = f\omega^n$ $\quad \sup \varphi = 0$, $\omega + i\partial\bar\partial\varphi \geq 0$

provided $\int_V f\omega^n = \int_V \omega^n$

Thm: if f has "algebraic singularities" then φ smooth outside $\mathrm{Sing}(V) \cup \mathrm{sing}(f)$

Main Thm: requires adaptation to singular context of Kolodziej 1996. pluripotential approach to Yau's C^0-~~est~~ estimate

Reg Thm: follows from technique of Tsuji, 1988

§ 3. Singular ~~K~~E metrics

1. X canonical 1-Gorenstein $\alpha \in K_{X,x}$

$\alpha \in \Omega^n(X_x^{\text{smooth}})$ $\quad V(\alpha) = \varepsilon_n \alpha \wedge \bar\alpha$

X canonical $\Rightarrow \int_{X_z^{\text{smooth}}} \varepsilon\, \alpha \wedge \bar\alpha < \infty \Rightarrow \varepsilon\, \alpha \wedge \bar\alpha$ extends as a measure with algebraic singularity.

(V, Δ) klt projective $\quad K_X + \Delta \sim_{\mathbb{Q}} 0$, $\alpha \in H^0(V; \mathcal{O}_V(m(K_V + \Delta)))$

weak solution of $(\omega + i\partial\bar\partial\varphi)^n = \varepsilon\, \alpha \wedge \bar\alpha$. it's smooth outside the singularities of (V, Δ)

on $(V_\Delta^{\text{smooth}}$, defines a Ricci flat metric.

Thm: same with $K_X + \Delta > 0$ (we need finite generalization)