NONCOMMUTATIVE MODULI SPACES OF VECTOR BUNDLES

Thm: [Narasimhan - Ramanan 74]

X smooth curve of genus $g \geq 2$

M moduli space of stable vector bundles on X w/ fixed determinant of rank r & degree d.

Assume $(r,d)=1$ ($\Rightarrow$ M smooth & proper)

Then

$$\mathrm{Def}(X) \xrightarrow{\sim} \mathrm{Def}(M)$$

If X is a surface & M moduli space of bundles on X, then there is not even a map

$$\mathrm{Def}\, X \longrightarrow \mathrm{Def}\, M$$

in general, because a vector bundle E/X need not extend to a deformation.

e.g. if X K3, $c_1(E) \neq 0$, $\exists$ deformation of X s.t. E does not extend.

Remark: OK if $\mathrm{Ext}^2(E,E) = 0 \ \forall\ [E] \in M$,
e.g. if X del Pezzo, E stable.

In general, $\mathrm{Ext}^2(E,E) = H^2(\mathrm{End}^0 E) \oplus H^2(\mathcal{O}_X)$

expect zero if M smooth usually nonzero

<u>Thm</u>: X K3 surface

M moduli space of stable vector bundles on X.

Assume M compact, $\dim M = 2$

($\Rightarrow$ M K3 [Mukai])

Then

$$\mathrm{NCDef}\, X \xrightarrow{\sim} \mathrm{NCDef}\, M$$

Here $\mathrm{NCDef}\, X$ denotes the space of noncommutative deformations of X in the following sense.

X smooth variety $/k$

A Artinian local ring, $A/m = k$.

A NC deformation of X over A is:

0) $X = \cup U_i$

1) $\mathcal{O}_{V_i}/U_i$ flat associative A-algebra

w/ $\phi_i : \mathcal{O}_{V_i} \otimes k \xrightarrow{\sim} \mathcal{O}_{U_i}$

2) $\psi_{ij} : \mathcal{O}_{V_i}|_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{V_j}|_{U_{ij}}$ compatible w/ ϕ_i

3) $c_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_{V_k}^\times)$ s.t. $\bar{c}_{ijk} = 1 \in \Gamma(U_{ijk}, \mathcal{O}_{U_k}^\times)$

s.t. • $\psi_{jk} \circ \psi_{ij} = \mathrm{ad}(c_{ijk}) \circ \psi_{ik} = c_{ijk}\, \psi_{ik}(-)\, c_{ijk}^{-1}$

• $c_{jkl}\, c_{ijl} = \psi_{kl}(c_{ijk})\, c_{ikl}$

Equivalence: $(\mathcal{O}_{V_i}, \psi_{ij}, c_{ijk}) \sim (\mathcal{O}_{V_i}, \mathrm{ad}\, b_{ij} \circ \psi_{ij}, c'_{ijk})$

where $c'_{ijk} = b_{jk}\, \psi_{jk}(b_{ij})\, c_{ijk}\, b_{ik}^{-1}$

Note: $\mathcal{O}_{V_i}$ do not glue to give a sheaf on X unless c_{ijk} central.

However, in our setting, we may assume $c_{ijk} \in A^\times$ (reason: $H^2(X,\mathbb{C}) \twoheadrightarrow H^2(X,\mathcal{O}_X)$)

Then deformation given by

1) $\mathcal{O}_Y / X$ flat associative A-algebra w/ $\phi : \mathcal{O}_Y \otimes k \xrightarrow{\sim} \mathcal{O}_X$

2) $c_{ijk} \in A^\times$ s.t. $\bar{c}_{ijk} = 1 \in k^\times$ and

$$c_{jkl}\, c_{ijl} = c_{ijk}\, c_{ikl}$$

(i.e. c a 2-cocycle).

Conceptual explanation: [Kontsevich, Van den Bergh – Lowen]

we are deforming the <u>category</u> $\mathrm{Coh}(X)$.

$$\mathrm{Coh}(\mathcal{O}_{U_i}) \rightsquigarrow \mathcal{C}_i = \mathrm{Coh}(\mathcal{O}_{V_i})/U_i$$

coherent left $\mathcal{O}_{V_i}$-modules

$$G_{ij} : \mathcal{C}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{C}_j|_{U_{ij}}$$

functor (equiv of cats)

$$\begin{tikzcd} \mathcal{C}_i|_{U_{ijk}} \arrow[r, "G_{ij}"] \arrow[rr, bend right, "G_{ik}"'] & \mathcal{C}_j|_{U_{ijk}} \arrow[d, phantom, "\Downarrow n_{ijk}"] \arrow[r, "G_{jk}"] & \mathcal{C}_k|_{U_{ijk}} \\ & {} & \end{tikzcd}$$

natural isom

such that

$$\begin{tikzcd} & j \arrow[r] \arrow[d, phantom, "\Downarrow"] & k \arrow[d] \\ i \arrow[ur] \arrow[urr] \arrow[rr] & {} \arrow[ur, phantom, "\Downarrow"] & l \end{tikzcd} = \begin{tikzcd} & j \arrow[r] \arrow[dr] \arrow[d, phantom, "\Downarrow"] & k \arrow[d] \arrow[dl, phantom, "\Downarrow"] \\ i \arrow[ur] \arrow[rr] & {} & l \end{tikzcd}$$

$\rightsquigarrow$ category $\mathcal{C}/X$.

[Van den Bergh - Lowen 04, 05]

Every deformation of $\mathrm{Coh}(X)$ is obtained in this way.

In terms of cocycles:

$$\mathrm{Ob}\,\mathcal{C} = \quad \mathcal{F}_i / U_i \quad \text{left } \mathcal{O}_{V_i}\text{-module}$$

$$g_{ij} : \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j|_{U_{ij}}$$

$$\text{s.t.} \quad g_{jk} \circ g_{ij} = c_{ijk} \cdot g_{ik}$$

"twisted sheaves"

Hochschild cohomology

Hochschild-Kostant-Rosenberg

$$HH^*(X) := \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \bigoplus_{p+q=*} H^p(\Lambda^q T_X)$$

__Thm:__ X smooth proper variety

- 1st order NC defs of X classified by $HH^2(X)$.

Let $0 \to I \to A' \to A \to 0$ be extension of Artinian local rings w/ $m.I = 0$, &

$$\begin{array}{ccc} X & \subset & Y \\ \downarrow & & \downarrow \\ \mathrm{Sp}\,k & \subset & \mathrm{Sp}\,A \end{array}$$

NC def of X over A

- $\exists$ extension of Y over A' iff obstruction $o \in HH^3(X) \otimes I$ vanishes
- If $o = 0$, extensions classified by $HH^2(X) \otimes I$.

e.g. $$HH^2(X) = \underbrace{H^0(\Lambda^2 T_X)}_{\text{bracket}} \oplus \underbrace{H^1(T_X)}_{\text{comm. def}} \oplus \underbrace{H^2(\mathcal{O}_X)}_{\text{gerbe}}$$

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_Y \to \mathcal{O}_X \to 0$$

$$[\,,\,] : \mathcal{O}_X \times \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

$$x, y \longmapsto \tilde{x}\tilde{y} - \tilde{y}\tilde{x}$$

$$0 \to \mathcal{O}_X \longrightarrow \mathcal{O}_Y^{\times} \longrightarrow \mathcal{O}_X^{\times} \to 1$$

$$f \longmapsto 1 + tf$$

$$c_{ijk} = 1 + t\alpha_{ijk} \qquad (\alpha_{ijk}) \in H^2(\mathcal{O}_X)$$

$$HH^3(X) = H^0(\Lambda^3 T_X) \oplus H^1(\Lambda^2 T_X) \oplus H^2(T_X) \oplus H^3(\mathcal{O}_X)$$

Jacobi identity

$\Updownarrow$

associativity

comm.

obstructions

Also $HH^1(X) = H^0(T_X) \oplus H^1(\mathcal{O}_X)$ = infinitesimal autos.

- $\mathrm{Aut}(Coh(X)) = \mathrm{Pic}\,X \rtimes \mathrm{Aut}\,X$,

$\mathcal{F} \mapsto \mathcal{F} \otimes L$

Ex: X K3. $HH^1(X) = HH^3(X) = 0$,

$$HH^2(X) = \underset{1}{H^0(\Lambda^2 T_X)} \oplus \underset{20}{H^1(T_X)} \oplus \underset{1}{H^2(\mathcal{O}_X)}$$

$\Rightarrow$ NC Def(X) smooth, universal, dim 22

$\because$ no infinitesimal autos

Remark: NC deformations should be related to defs of X as generalized complex manifold [Hitchin, Gualtieri] – also computed by $HH^*(X)$

Non commutative moduli of vector bundles

X smooth proper variety

M moduli space of stable vector bundles on X

(assume smooth, proper).

Suppose given $\begin{array}{c} Y \\ \downarrow \\ \operatorname{Sp} \Lambda \end{array}$ NC def of X over

complete local ring Λ.

We construct $\begin{array}{c} N \\ \downarrow \\ \operatorname{Sp} \Lambda \end{array}$ induced NC thickening

of M as <u>noncommutative moduli space</u>.

$$N \longrightarrow (NC/\Lambda)^{opp}$$

NC/Λ := category of NC-complete associative Λ-algebras R [Kapranov 98]

i.e. $R = \varprojlim R/F^{\wedge}R$

$F^{\wedge}R$ = 2 sided ideal gen'd by expressions involving $\wedge$ commutators.

e.g. $F^1R = ([R,R])$, $R/F^1R = R_{ab}$, abelian-isation

$F^2R = ([R,[R,R]], [R,R].R.[R,R])$, etc.

$N(R) := \{$ locally free left $\mathcal{O}_Y \otimes_\Lambda R$-modules $F \}$

Morphisms: $F' \to F$ over $R' \leftarrow R$ is isom

Then N is a stack over $(NC/\Lambda)^{opp}$ (with Étale topology).

Moreover it is "algebraic": versal deformations exist, and isoms are representable.

Local model:

$$R \rightrightarrows R *_{\Lambda} \Lambda[t, t^{-1}] \qquad [\text{"}\operatorname{Spec} R\text{"}/\mathbb{G}_{m,\Lambda}]$$

$$r \longmapsto t r t^{-1} \qquad \text{acting by conjugation.}$$

where R is base of versal def.

<u>Note</u>: Although $\mathbb{G}_m = \operatorname{Aut}(E)$ acts trivially on commutative def space, action is nontrivial on NC Def: mult. by scalar is <u>not</u> hom of

left R-mods if R NC

[Schlessinger 68]

Art $/\Lambda$ = Artinian local Λ-algebras A, $A/m = k$

F: Art $/\Lambda \longrightarrow$ Sets functor.

Conditions for F to admit versal deformation, R/Λ, $\xi \in F(R)$. $\left(F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''), \right.$

Construction: $x_1, \dots, x_n$ basis of tangent space $t_F = F(k[t]/(t^2))$.

$R = \Lambda[\![x_1, \dots, x_n]\!]/J$ where J minimal ideal s.t. $\exists$ lift of 1st order def to R.

Exactly same argument works for

F: NC-Art $/\Lambda \longrightarrow$ Sets

$R = \Lambda\langle\langle x_1, \dots, x_n \rangle\rangle / J$, J 2-sided ideal.

Atiyah class [Atiyah57]

$$0 \to \mathcal{O}_X \to \mathcal{D}_X^{\leq 1} \xrightarrow[\text{symb}]{} T_X \to 0$$

$( )^* \otimes E \quad \Big\downarrow$

$$0 \to \Omega_X \otimes E \to (\mathcal{D}_X^{\leq 1})^* \otimes E \to E \to 0$$

$a(E) \in \mathrm{Ext}^1(E, \Omega_X \otimes E)$, extension class.

<u>Prop</u> [Illusie71]

$$\begin{array}{ccc} X & \subset & Y \\ \downarrow & & \downarrow \\ \mathrm{Sp}\,k & \subset & \mathrm{Sp}\,k[t]/(t^2) \end{array}$$

1st order def. corr. to $u \in H^1(T_X)$.

Then E extends to Y iff

$$u \times a(E) = 0 \in \mathrm{Ext}^2(E, E)$$

Prop [Y. Toda 05]

$$\begin{array}{ccc} X & \subset & Y \\ \downarrow & & \downarrow \\ \operatorname{Spk} & \subset & \operatorname{Sp} k[t]/(t^2) \end{array}$$

NC def corr to $u \in HH^2(X)$.

Then E extends to Y iff

$$\Big(u \times \underset{\substack{\| \\ 1 + a(E) + \frac{1}{2} a(E)^2 + \cdots}}{\exp(a(E))} \Big)_2 = 0 \in \operatorname{Ext}^2(E, E)$$

Simple minded proof:

Fix trivialisations $\varphi_i : E|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^{\oplus r}$,

& write $\varphi_j \circ \varphi_i^{-1} = (-).g_{ij}$, $g_{ij} \in GL_r(\mathcal{O}_{U_{ij}}$

so $g_{ij} g_{jk} g_{ik}^{-1} = 1$.

Pick lifts $\tilde{g}_{ij}$ & compute $\tilde{g}_{ij} \cdot \tilde{g}_{jk} \cdot \tilde{g}_{ik}^{-1}$ .. $\overline{L}$

X smooth proper variety.

E vector bundle on X

Y/Λ NC deformation of X/k.

Prop. Assume
- $H^2(\mathrm{End}^0 E) = 0$
- $\dim H^1(\mathrm{End} E) = 2$
- $\Lambda^2 H^1(\mathrm{End} E) \xrightarrow[\mathrm{tr}(\cdot \cup \cdot)]{\sim} H^2(\mathcal{O}_X)$

Let R be hull for $\mathrm{Def}E : \mathrm{Art}/k \longrightarrow \mathrm{Sets}$

S/Λ " " " $\mathrm{NCDef}E : \mathrm{NC\text{-}Art}/\Lambda \longrightarrow \mathrm{Sets}$.

Then S flat over Λ & $S \otimes k = R$.

Ex: X K3.

$$\Lambda^2 H^1(\mathrm{End} E) \longrightarrow H^2(\mathcal{O}_X) \cong \mathbb{C}$$

non degenerate (by Serre duality)

$\rightsquigarrow$ hol. symplectic form on moduli space M.

So OK if $\dim M = 2$

Sketch proof:

At each order, can kill obstruction to lifting universal bundle by choosing appropriate NC deformation of R, because

$$\Lambda^2 T_{R,0} \xrightarrow{\sim} H^2(\mathcal{O}_X) = H^2(\mathrm{End}\,E) = \text{obs. space}$$

Note also R smooth & $\dim R = 2 \Rightarrow \Lambda^3 T_R = 0$
$\Rightarrow$ NC Defs of R unobstructed.

Injectivity of $(\dagger) \Rightarrow$ special fibre commutative. $\square$

<u>Ex2</u>: $X = V/L$ complex torus

$$M = \mathrm{Pic}^0 X = \bar{V}^*/L^*$$

(here need to show $\nexists$ obstructions in $\Lambda^3 T_M$ for $\dim M > 2$) NC BTT ?

[Ben-Bassat — Block — Pantev 05]

'purely Poisson' def. of X

Can get global description of N by repeating construction globally:–

X, M K3 s, $F_0 / X \times M$ univ. bundle,

Y / Λ NC def of X.

Lift F_0 to $F / Y \times_\Lambda N$.

Obstruction $\quad HH^2(M) \xrightarrow{\sim} Ext^2(F_0, F_0)$

$$p_M^*(-) \times \exp(a(F_0))$$

$$\begin{array}{ccc} a(F_0) & \in & H^1(\Omega_{X \times M} \otimes End\, E) \\ \| & & \| \\ a(F_0) \oplus e_X \oplus e_M & \in & H^1(\Omega_{X \times M}) \oplus H^0(\Omega_X \otimes R^1_{p_{X*}} End\, E) \\ & & \oplus H^0(\Omega_M \otimes R^1_{p_{M*}} End\, E) \end{array}$$

Here $e_M : T_M \longrightarrow R^1 p_{M*} End\, E$

is Kodaira-Spencer map.

$$HH^2(M) \xrightarrow{\sim} \mathrm{Ext}^2(F_0, F_0)$$

$$\begin{array}{c|ccc} & H^2(\mathcal{O}_M) & H^1(T_M) & H^0(\Lambda^2 T_M) \\ \hline H^2(\mathcal{O}_M) & 1 & c_1(F_0|_{\cdot \times M}) & ch_2(F_0|_{\cdot \times M}) \\ H^1(R^1 p_{M*} \mathrm{End}\, E) & 0 & H^1(e_M) & \cdot\cdot \\ H^2(\mathcal{O}_X) & 0 & 0 & \mathrm{tr}(e_M \wedge e_M) \end{array}$$

symplectic form

The composition

$$\begin{tikzcd} HH^2(X) \arrow[r, "\sim", "- p_X^*(-) \times \exp(a(F_0))"'] \arrow[rr, bend right] & \mathrm{Ext}^2(F_0, F_0) & HH^2(M) \arrow[l, "\sim"', "p_M^*(-) \times \exp(a(F_0))"] \end{tikzcd}$$

gives the map of 1st order defs.
The summands of HH^2 are thoroughly mixed.

<u>Note</u>: Y. Toda studied correspondence of 1st order NC defs in general setting (when $D(X) \xrightarrow{\sim} D($