Reconceptualizing Mathematics

Course Materials for Elementary and Middle School Mathematics

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Underlying Assumptions

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- Such knowledge is best developed through reasoning, conjecturing, discussing, and explaining mathematics. A persistent pursuit of explanation is a hallmark of a classroom in which learning is taking place.

Underlying Assumptions

- Teachers must have a deep, profound knowledge of the mathematics they will teach.
- Such knowledge is best developed through reasoning, conjecturing, discussing, and explaining mathematics.
- This development is more likely to happen in classes and professional development during which good pedagogy is modeled. Prospective teachers are unlikely to demonstrate flexible, interactive styles of teaching unless they have experienced mathematics taught this way.

CBMS recommends that the preparation of mathematics teachers should include courses that develop

- “a deep understanding of the mathematics they teach,
- careful reasoning and ‘common sense’ in analyzing conceptual relationships,
- the habits of mind of a mathematical thinker and that demonstrate flexible, interactive styles of teaching.”

(The Mathematical Preparation of Teachers, 2001, pp. 7-8).

Future teachers learn mathematics when they themselves experience doing mathematics--

“exploring, guessing, testing, estimating, arguing, and proving--in order to develop confidence that they can respond constructively to unexpected conjectures that emerge as students follow their own paths in approaching mathematical problems.”

(NRC, 1989, p. 65)

“The mathematics needed by prospective middle-grades teachers encompasses the mathematics needed by teachers in the lower grades, but extended in several important ways to reflect the more sophisticated mathematics curriculum of the middle grades.”

CBMS, 2001, p. 99.

Thus we provide many forms of instructional assistance intended

--to help instructors better appreciate the nontrivial nature of the school mathematics prospective teachers need to know, and

--to begin to model teaching strategies that these prospective teachers will be expected to use in their own classrooms.

Discussion, Is One Way Better than Another? The methods used by the different children deserve a class discussion. Students often don't appreciate good estimation skills and need to reflect on what is going on in each example. Discuss with students the relative sophistication of these methods. For example, Shawn rounds both numbers up—forcing the estimate to be too high. There is no compensation made, such as saying "It will be less than 2400."

See INSTRUCTOR NOTE 5.2 for additional information on these children's methods.

Discussion: Is One Way Better Than Another?

1. Estimate 36×55 alone before discussing this problem.
2. Carefully read these students' solutions to the first exercise. Be sure you understand the thinking behind each one. Then discuss them in terms of whether each way is a good way to estimate. Did you use one of these ways?

   Shawn: Round to 40 and 60. $40 \times 60 = 2400$.

   Jack: First round down: $30 \times 50 = 1500$. Then round up: $40 \times 60 = 2400$. So it's about in the middle, maybe a little past. So I'd say 2000.

   Maria: Rounding both up would make it too big, so I'll round 36 to 40 and 55 to 50. $40 \times 50 = 2000$.

   Jimmy: A little more than 36×50, which is $36 \times 100 \div 2$ and that's $18 \times 100 = 1800$. It's about 5×36 more, or about 180 more, so I'll say 1980.

   Deb: Rounding both up gives $40 \times 60 = 2400$. Since that's too big, I'll say it's about 2200.

   Sam: A little more than $6 \times 6 \times 50$, which is $6 \times 300 = 1800$. So I'll say 1900.

INSTRUCTOR NOTE 5.2

Here are some comments that you might hear from students, or want to bring into the discussion.

- Jack's way is easy, efficient, and gives a better estimate than Shawn's way (unless the context calls for an over-estimate).
- Maria uses a safe, reliable method of rounding one factor up, the other down.
- Jimmy shows good number sense than helps him compensate to what is the exact answer.
- Deb shows what Shawn should have done—compensate for his rounding.
- Sam's way is mathematically appealing—he is thinking outside the box.

There are four parts to RM:

1. Reasoning About Numbers and Quantities
   (a little more than a semester of work)

2. Reasoning About Algebra and Change
   (a little less than a semester of work)

3. Reasoning About Shapes and Measurement
   (about a semester of work)

4. Reasoning About Chance and Data
   (about a semester of work)

Materials can be organized for different courses.

What about problem solving?

Problem solving is “not only a goal of mathematics but a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and then be encouraged to reflect on their thinking.”

NCTM, 2000, p. 52

Problems permeate the texts. The activities, discussions, and learning exercises focus, for the most part, on problems, some of them quite challenging. In each part there is explicit emphasis on problem solving. For example,

- In RNQ problems are solved by analyzing their quantitative structure.

- In RAC, being able to describe graphs qualitatively is fundamental to the notion of change.

- In RSM there are rich problems that explore and develop spatial reasoning.

- In RCD some of the misconceptions about probability are explored.

The Role of Technology--Each of the four parts uses technology in different ways. For example...

- In RNQ, appropriate use of calculators is discussed. Learning exercises call on the use of Internet applets. Video clips of student learning are used.

- In RAC, motion detectors are highly recommended and an applet focusing on average rate of change is used.

- In RSM, an optional unit on Geometer’s Sketchpad was designed and is coordinated with the text.

- In RCD, faculty and students can choose between Fathom, the TI-73, or Excel to analyze data. On-line applets are also recommended.

The Intent of the Learning Exercises...

The format and content of the end-of-section learning exercises are somewhat different from what prospective teachers may have experienced in previous mathematics courses. The primary purposes of these exercises is to provide students with a deeper understanding of the content of that section *as well as* gain needed mathematical skills.

The role of explaining--

Students are told that the overall goal of these courses is to come to understand the mathematics deeply enough to participate in meaningful conversations about this mathematics and its application, and that being capable of solving a problem or performing a procedure, by itself, will not enable them to add value to the school experience of their students.

Sense-making is a paramount objective.

Key Ideas in Reasoning About Numbers

- Our numeration system can be understood only if the significant role of place value is recognized.
- Different generic contexts can be modeled by arithmetic operations, and hence there are different ways of thinking about these operations.
- *Student-invented nonstandard algorithms can be used to help them understand traditional algorithms.
- *Mental computation, estimation, paper-and-pencil calculations, and calculators are appropriate at different times.
- Facility in changing easily from one number representation to another is fundamental to having good number sense.
- The mathematics underlying algorithms for operating on whole numbers and on rational numbers can and should be understood.

Activity: Children's Ways

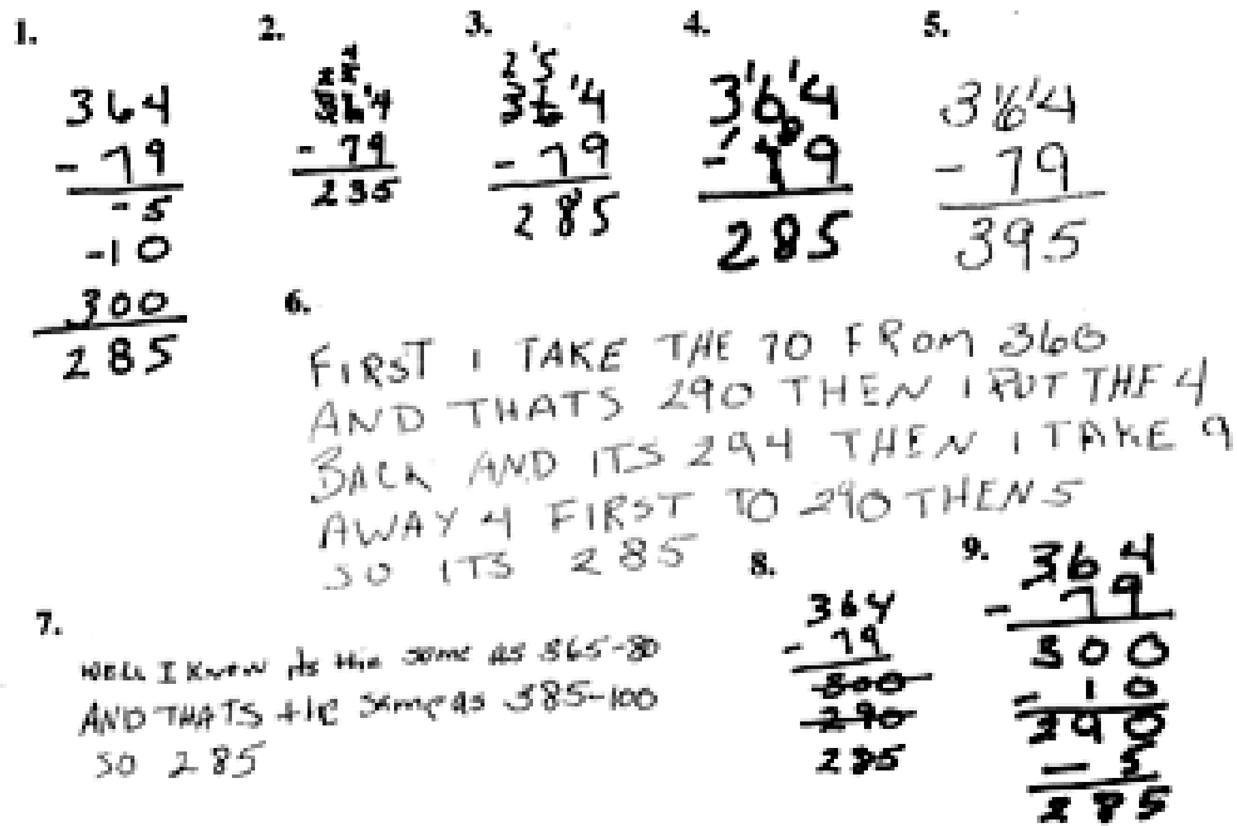
1. Consider the work of six second-graders, all solving 364 – 79 (in written form, without calculators or base ten blocks).

   Identify:

   a. which students clearly understand what they are doing;

   b. which students might understand what they are doing; and

   c. which students do not understand what they are doing.



1.

```
  364
-  79
   -5
  -10
  300
  285
```

2.

```
  364
-  79
  235
```

3.

```
  364
-  79
  285
```

4.

```
  364
-  79
  285
```

5.

```
  364
-  79
  395
```

6.

FIRST I TAKE THE 70 FROM 360
AND THATS 290 THEN I PUT THE 4
BACK AND ITS 294 THEN I TAKE 9
AWAY 4 FIRST TO 290 THEN 5
SO ITS 285

7.

WELL I KNEW its the same as 365-80
AND THATS the same as 385-100
SO 285

8.

```
  364
-  79
  300
  290
  285
```

9.

```
  364
-  79
  300
  -10
  290
   -5
  285
```

Key Ideas in Reasoning About Quantities

- *Mathematical problem situations can be better understood when they are analyzed by recognizing the quantities involved and the relationships among these quantities.
- Understanding quantities includes understanding the values a quantity can take on.
- *Quantities can be compared additively or multiplicatively.
- A multiplicative comparison of two quantities is a ratio, which is itself a quantity.
- A ratio can be thought of as a measure of some attribute.
- A rate is a ratio of quantities that change, but without changing the value of the ratio. Equal ratios represent the same rate.

What do we mean by Quantitative Reasoning?

- A **quantity** is anything (an object, event, or quality thereof) that can be measured or counted. For example, the time it takes to drive from point A to point B; the number of spoons on the table, etc.

- The **value** of the quantity is its measure or count. For example, the driving time is 3 hours, there are 8 spoons on the table, etc.

Undertaking a Quantitative Analysis of a Problem

To **understand a problem situation** means understanding the quantities embedded in the situation and how they are related to one another.

The first step in undertaking a quantitative analysis of a problem is to identify the quantities involved in the situation (without, at this point, considering the values of quantities). Then ask: how the quantities are related to one another? A diagram is often very useful in developing an understanding of the situation.

Note that some quantities may not be explicitly stated although they are essential to the ***quantitative structure*** of the situation.

Example: Solve this problem quantitatively:

Dieter A: “I lost 1/8 of my weight. I lost 19 pounds.”

Dieter B: “I lost 1/6 of my weight. And now you weigh 2 pounds less than I do.”

What was B’s weight before the diet?

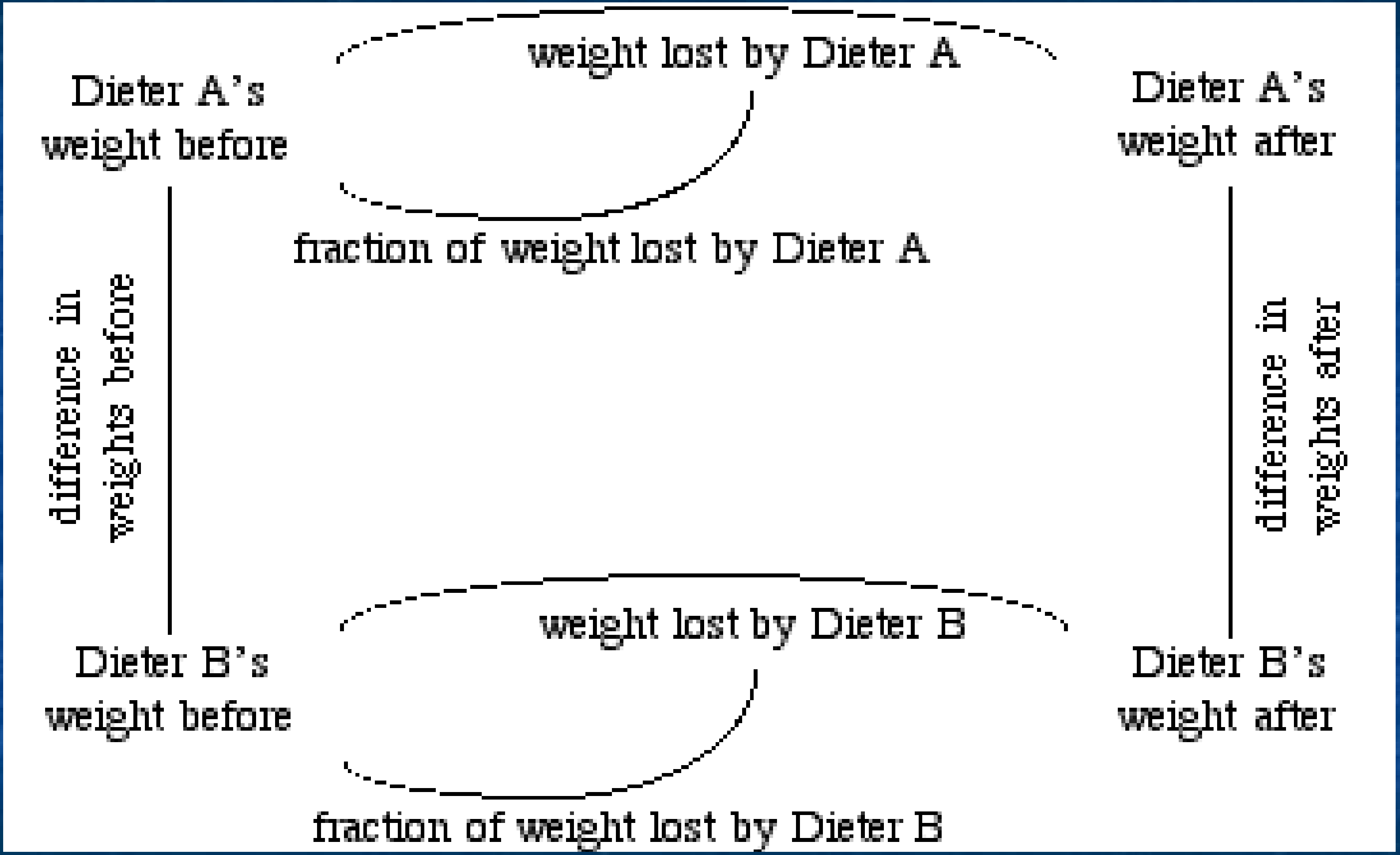
What quantities are involved here?

- Dieter A's weight before the diet.
- Dieter A's weight after the diet.
- Dieter B's weight before the diet.
- Dieter B's weight after the diet.
- The fraction of the weight lost by Dieter A.
- The fraction of the weight lost by Dieter B.
- The amount of weight lost by Dieter A.
- The amount of weight lost by Dieter B.
- The difference in their weights before the diet.
- The difference in their weights after the diet.

Note that the values of the quantities are not yet considered.

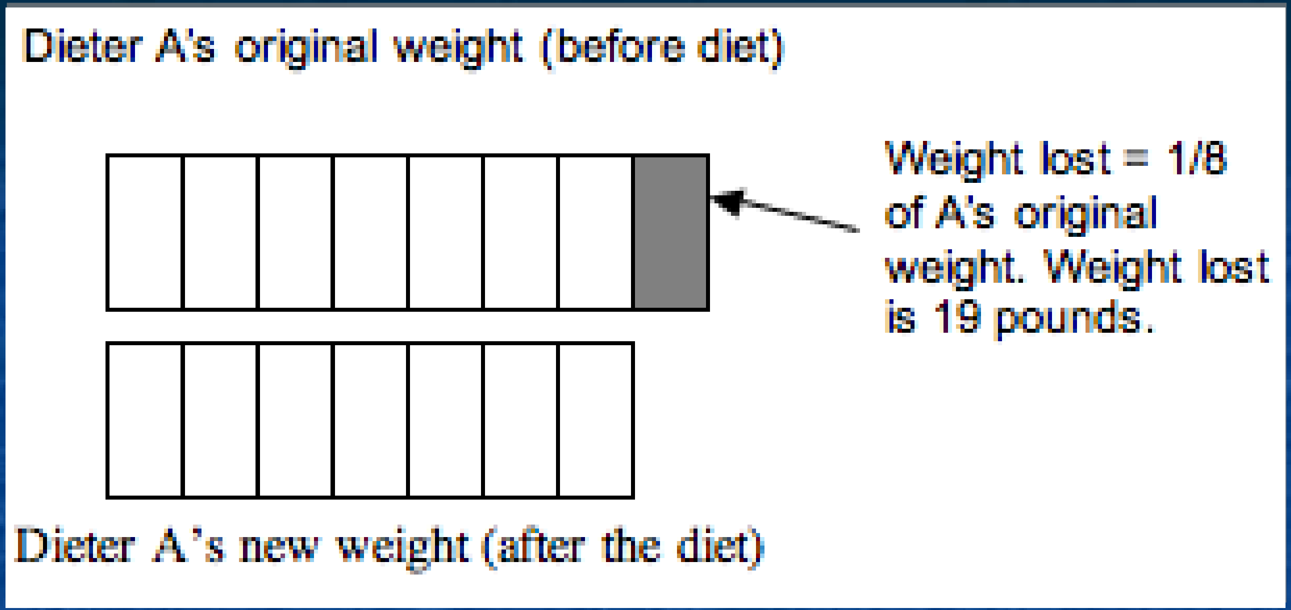
How are the quantities related?

A key to understanding the Dieters' Problem is to realize that there are two different ways to compare weights of the dieters before and after the diet. One can either consider the *difference* of weights lost, or the *ratio* of weights lost.


weight lost by Dieter A
Dieter A's weight before
Dieter A's weight after
fraction of weight lost by Dieter A
difference in weights before
difference in weights after
weight lost by Dieter B
Dieter B's weight before
Dieter B's weight after
fraction of weight lost by Dieter B

What quantities are involved here?

- Dieter A's weight before the diet.
- Dieter A's weight after the diet.
- Dieter B's weight before the diet. **???**
- Dieter B's weight after the diet.
- The fraction of the weight lost by Dieter A. **1/8**
- The fraction of the weight lost by Dieter B. **1/6**
- The amount of weight lost by Dieter A. **19 lbs.**
- The amount of weight lost by Dieter B.
- The difference in their weights before the diet.
- The difference in their weights after the diet. **2 lbs. (A is 2 lbs. less than B.)**



- 8 x 19 is 152 which is A's weight before the diet.
- 152 – 19 = 133 which is A's weight after the diet.
- 133 + 2 = 135 is B's weight after the diet.
- 135 is 5/6 of B's weight before the diet.
- So 1/6 of B's weight before the diet is 27 pounds.
- **B weighed 27 x 6 = 162 pounds before the diet.**

In summary then, what do we mean by Quantitative Reasoning?

- Situations can be better understood when they are analyzed by recognizing the quantities involved and the relationships among these quantities.

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- Situations can be better understood when they are analyzed by recognizing the quantities involved and the relationships among these quantities.
- Quantities can be compared additively or multiplicatively, depending on the nature of the situation. Students should recognize whether a situation is additive or multiplicative and know how to respond.

In summary then, what do we mean by Quantitative Reasoning?

- Situations can be better understood when they are analyzed by recognizing the quantities involved and the relationships among these quantities.
- Quantities can be compared additively or multiplicatively, depending on the nature of the situation.
- A multiplicative comparison of two quantities results in a ratio, which is itself a quantity, and which can be thought of as a measure of some attribute.

When solving a word problem (such as the dieters problem):

Instead of playing a guessing game by asking “What operations do I need to perform, with what numbers, and in what order?” one would be better off asking questions such as these:

What do I know about this situation?

What quantities are involved here? Which ones are critical?

Which quantities do I know the value of?

Which quantities do I not know the value of? Do I need to know any of these values?

Are there any quantities that are related to others? If so, how are they related? Can these relationships enable me to find any unknown values?

Key ideas in Reasoning About Algebra and Change

1. Symbols are used in algebra to help us represent quantities and to generalize patterns.
2. *Quantitative reasoning about problem situations with numbers translates readily into reasoning about problem situations with symbols.
3. Tables, graphs, and equations are three fundamental ways of exhibiting information about things that change, and one should be able to move effortlessly from one to another.
4. *The slope of a straight line provides information about rate of change, including average rate of change.
5. *Graphs without numbers (qualitative graphs) are useful in coming to understand how to interpret graphs.
6. Algebra can be thought of as a language, a generalized quantitative reasoning, a generalized arithmetic, and a tool for problem solving.
7. Numerical patterns can often be generalized, using algebra.

Studying Change: Consider this story...

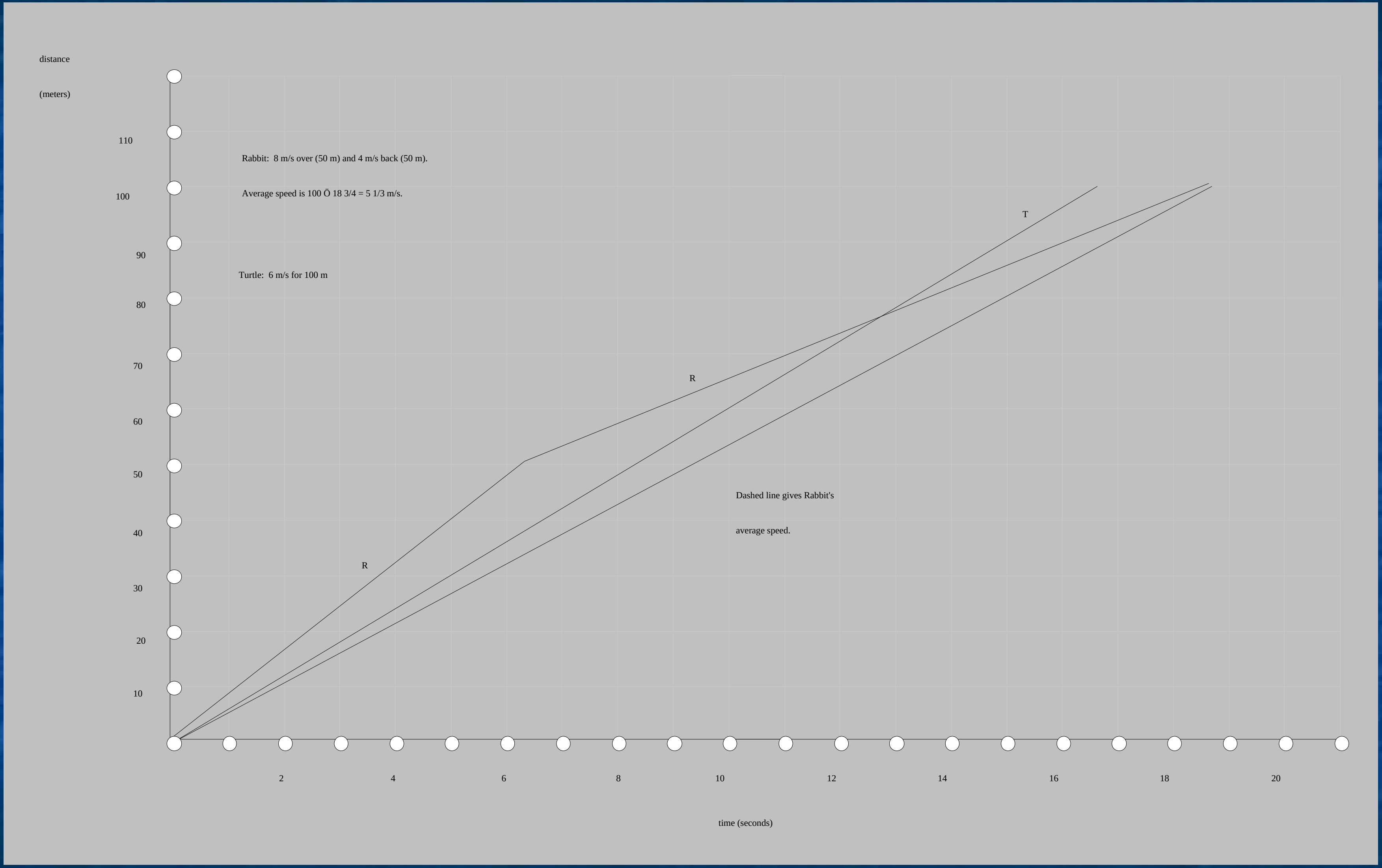
To Angie's House and Back

Rob the rabbit decided to go visit his friend Angie who lived 50 meters away in another rabbit hole. He walked the 50 meters over at a steady pace of 8 meters per second. When he returned, he was tired and walked the 50 meters home at only 4 meters per second. At the same time, Rob's roommate Sam the Turtle also walked from Rob's hole to Angie's hole and back at a constant rate of 6 meters per second. Who arrived back home first?

Many say that they arrive home at the same time; that the average speed for both was 6 meters per second. Is this correct?

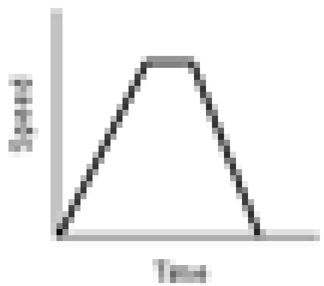
But consider these questions:

- How long did it take Robert to get to Angie’s hole?
- How long did it take Robert to get home?
- What was his total time walking?
- What was the total distance he walked?
- What was Robert’s average rate (speed) for the entire trip? (Hint: What is 100 meters divided by the total time?)
- What was Stuart’s average rate (speed)?
- Who arrived home first?

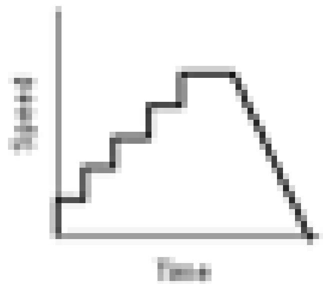

distance
(meters)
110
100
90
80
70
60
50
40
30
20
10
Rabbit: 8 m/s over (50 m) and 4 m/s back (50 m).
Average speed is 100 Ö 18 3/4 = 5 1/3 m/s.
Turtle: 6 m/s for 100 m
T
R
R
Dashed line gives Rabbit's
average speed.
2
4
6
8
10
12
14
16
18
20
time (seconds)

A child climbs up a slide at a constant speed, stops at the top, and then slides down.

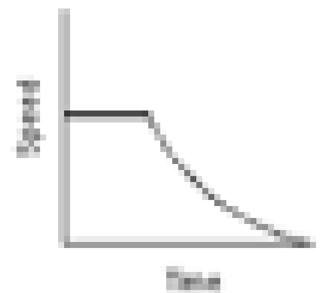
a.



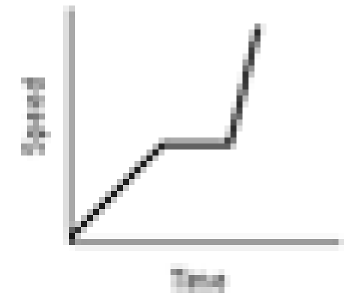
b.



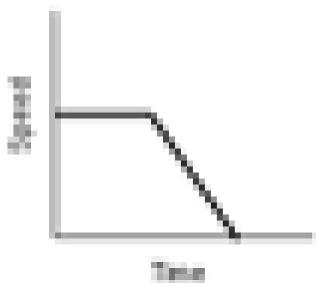
c.



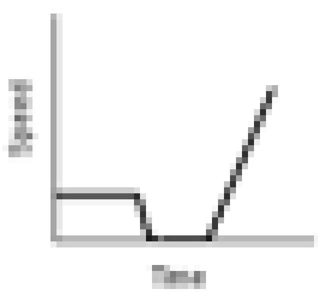
d.



e.



f.



Which graph best models the situation? For each of the other graphs, explain why it fails.

Key Ideas regarding Shapes

Beginning with a study of 3-dimensional shapes provides an entrance into unfamiliar shapes that motivate the study of geometry. Considering faces of polyhedra leads to a study of polygons.

*Exploring 3-dimensional shapes allows learning about spatial visualization and ways to practice it.

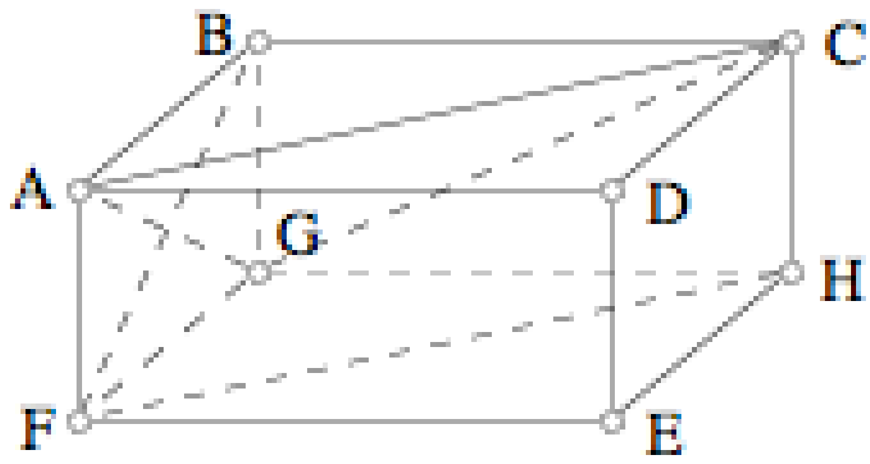
Shapes can be classified, giving rise to hierarchies. Students must understand and appreciate definitions of shapes and their names.

*Size changes in planar and space figures require understanding similarity (based on proportional reasoning). The use of a scale factor in determining size change leads to viewing size change as a multiplicative relationship.

Shapes, both 2- and 3-dimensional, may have symmetries of different types.

Rigid motions (isometries) include translations, reflections, rotations, and glide reflections. The figure that results from a rigid motion is congruent to the original figure.

18. Consider the right rectangular prism below:



a. Edge AF is perpendicular to (makes right angles with) which of these?

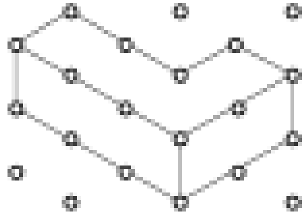
edge FE, or $\overline{FE}$ edge AD, or $\overline{AD}$ edge AB, or $\overline{AB}$

segment AG, or $\overline{AG}$ segment BF, or $\overline{BF}$ segment FH, or $\overline{FH}$

b. Name the endpoints of three edges that are parallel to edge AD.

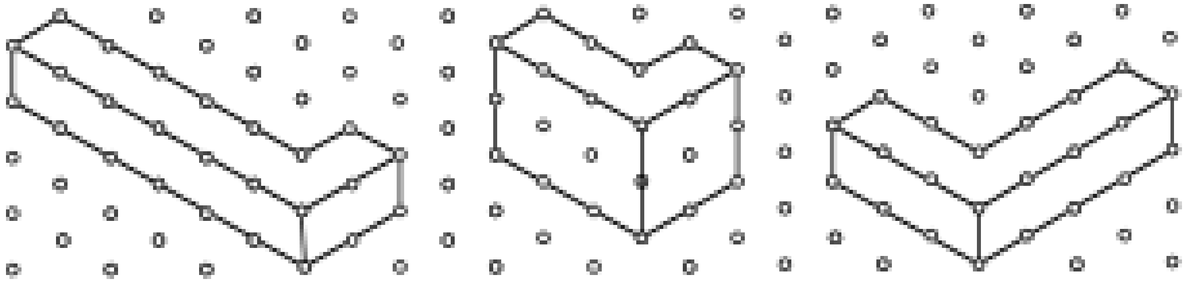
Discussion: Larger and Smaller

An eccentric 10^9-aire owns an L-shaped building like the one below.



She wants another building designed, "shaped exactly like the old one, but twice as large in all dimensions." Make or draw a model shaped like the building that she wants.

1. Below are some diagrams. Which, if any, of the following drawings will meet the criterion? Explain.



2. On isometric dot paper, show your version of a building (call it Building 2) that will meet the 10^9-aire's criterion, and compare your drawing with those of others. Discuss any differences you notice.

3. Now the 10^9-aire wants a Building 3 designed to be shaped like the original, but "three-fourths as large in all dimensions." Make a drawing or describe Building 3.

Elements for instructors

A research-based curriculum

A classroom-tested curriculum

Marginal notes and expanded notes

Answers to all learning exercises

Examination exercises in Word

Video clips and other examples of children's thinking

Appropriate technology incorporated into each part, with instructions on how to use the technology

For both instructors and students...

A message to students that lays out expectations of students as participants in learning;

A focus on sense-making;

Supplementary exercises with answers;

An appendix of review of “rules” of arithmetic that student are expected to know before the first course but can review here;

Discussion of NCTM content standards and Focal Points;

An inexpensively produced text to write in and keep as a reference.

A chapter structure that includes

Activities to be worked in groups (with marginal notes for instructors);

Discussions for the whole class (with marginal notes for instructors);

Definitions when appropriate; examples where needed;

‘Think Abouts’ intended to invite students to pause and reflect on what they have just read;

A Take-Away message at the end of each chapter summarizing major points in the chapter;

Learning Exercises that provide both challenge and practice;

Issues for Learning sections that examine what is known from research about the learning of one or more topics in the chapter;

Examples of children’s ways of doing mathematics.

It is difficult for a teacher to teach something which does not satisfy him entirely, but the satisfaction of the teacher is not the unique goal of teaching; one has at first to take care of what is in the mind of the student and what one wants it to become. (Poincaré)

For more information, go to

http://sdmp-server.sdsu.edu/sowder/

To view or review the course materials contact Steve Rigolosi at srigolosi@whfreeman.com

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