Connecting Advanced Mathematics to High School Content
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I will describe a course at Rutgers University called “Connecting Advanced Mathematics to High School Content”. To oversimplify the course was designed to answer that perpetual question from future math teachers: “Why must we learn stuff we will never teach in high school?” I will start by describing the context for the course, its goals, and its format. I want to spend the bulk of my time on sample connections from the course itself.

This course enrolls students in their fifth-year of our five-year B.A.-M.Ed.-Certification Program in mathematics. They will get a license good for K-12. Most of them will teach high school; some prefer grades 7-9. (We also have a three semester post-baccalaureate certification program. We have no purely undergraduate certification program at Rutgers.)

Our primary purpose is to give a substantial and useful answer to that persistent question from prospective high school math teachers, “Why must I learn material I will never teach?” A secondary purpose is to follow the recommendation in the CBMS report The Mathematical Education of Teachers for a capstone course to solidify and deepen understanding of relevant mathematical content for teachers.

We want to demonstrate that upper-level mathematics courses can
- Provide a deeper and more flexible understanding of the content taught in high schools
- Build the confidence in the correctness of material that is asserted without proof in high school
- Enhance teachers’ ability to cope with non-standard student ideas – by assisting students to identify circumstances in which these ideas are – or are not – valid.
We also want to reinforce students belief that it is easier
  • to recall information that makes sense, that is supported by
    evidence, and connects with other knowledge
  • to assert and use facts that are well understood.

The course meets once a week for a three hour period with a break in the middle. Students will have completed the full undergraduate mathematics major, a 30-credit minor in pedagogy, and a semester of practice teaching before enrolling in the Connections course.

The Five-year program requires fifth year students to take a math content course in addition to those used to satisfy the requirements of the math major. The Connections course is the recommended math course for this purpose. In 2006, ten of the 12 students in the cohort took it. In 2007, eight were enrolled on the first day – another eight added the course by the second class meeting.

In 2006 I team taught with Keith Weber of our GSE Math Ed group. This year I taught the course alone, since Keith was on leave. I am very grateful to Lara Alcock, now of University of Essex in England, for her help in designing and teaching earlier versions of the Connections course.

The course uses very little explicit exposition. We want to convince our students that they know more than they keep in quick-recall memory. To do this we show them ways to work out what they know but don’t recall easily. For this reason students usually work in small groups on what might be called “directed re-discovery tasks.” After maybe 20-30 minutes on each task, one or more groups present their work to the full class and we have a general discussion. Students submit a weekly written summary of the math from the previous week’s work. There is a midterm exam. Instead of a final exam, groups prepare written reports on projects and present oral summaries in class.
Let me now describe some of the connections we work on.

1. From Introduction to Mathematical Reasoning

Before this class students will have constructed some explicit proofs of results about rigid motions of the plane. We ask each group to solve equations like [slide]

\[(i) \quad X^2 - X + 6 = 0 \quad \text{or} \quad (ii) \quad X - \sqrt{X} + 6 = 0\]

and to present their results. Typically, what they present looks like this [slide]. With prompting they also check their results and discover a “spurious solution” in the second case. We then ask whether the spurious solution was the result of an algebraic error -- usually the students believe such an error occurred.

We then go back and turn the solution method into a “geometry style” deduction of a result. What we get looks something like this. [slide] This allows review of the differences between implications and equivalences and the difference between the logical connectives “and” and “or”. Students see that they had only deduced that

IF \( X \) is a real solution of (ii) THEN \( X = 9 \) or \( X = 4 \).

So there is no reason to be upset that 4 is not a solution, but that there is reason to check both possible values of \( X \) in the original equation. Most of the students seem to feel that the explicit reasoning makes the topic much less confusing. Sadly, they seem surprised that reasoning can help!

What if the quadratic in \( \sqrt{X} \) has two real solution or no real solutions? We also discuss these questions in detail to see what steps are not “reversible” and to understand the logical situation.
2. From Advanced Calculus (really baby real variables)

Our first intention is to demonstrate the importance of the completeness axiom by eliciting an explicit natural meaning for a non-terminating decimal expression as the least upper bound of the set of truncations of the expression.

Students find this dull in the abstract, but perk up with the questions of why

\[0.999\ldots9\ldots\] with an infinite string of 9’s

must represent a real number and why that real number must be 1.

We go on to some basic questions about numerical analysis. Suppose we know real numbers \(A\) and \(B\) accurately to \(n\) decimal places. What does that mean? How accurately can we compute \(A+B\), \(A\times B\), and \(A\div B\)? Students are led to recognize the connection between these questions and the proofs of the algebraic properties of limits of convergent sequences.

They are much more interested in questions such as “How accurately must you measure the dimensions of a dirty old billboard to compute its area within a pre-assigned allowable error?” This error estimate is exactly the one used to prove that the product of continuous functions is continuous. There are interesting continuations of the form “Suppose you know that a gallon of paint will cover between \(A\) and \(B\) square feet of dirty old bill board. How many gallons should the painter buy to ensure coverage and what is the maximum number of gallons he might then waste?”

It is interesting to me how uncomfortable these future teachers are with making estimates. They seem to have gotten the impression in their own high schools that the only applications of the axioms for an ordered field are those to “solving inequalities.”

They also seem to have the impression that estimation is the process by which you get an “exact” value from a calculator and then round off as directed. They were stunned that there were cases when one might have to use inaccurate data or that one might prefer to use less accuracy in order to simplify a computation.
3. From Introduction to Abstract Algebra (not very abstract)

We would like to demonstrate that there are good reasons to know and exploit the algebraic properties of common algebraic structures. We have worked extensively with various groups of rigid motions of the plane – such transformations get a lot of emphasis in the N.J. state curriculum standards. Here we have not been entirely happy with our choices of topics.

**Polynomials**

Instead we will probably try out some new exercises next year. We will begin with a careful discussion of the “factor root theorem”

Suppose $P(X)$ is a polynomial over the reals and $r$ is in $\mathbb{R}$. Then

$$r$$ is a root of $P(X)$ iff $(X-r)$ is a factor of $P(X)$.

The proof involves a case “division of polynomials.” We will go on to the so-called division algorithm, whose statement is not an algorithm, but whose proof is one. Here we would try to distinguish carefully between the result over a field (which allows any divisor) and the result over the integers (which allows only monic divisors). We would then specialize to the case of long division in the integers, where we represent these integers as polynomials in powers of 10, but allow only decimal digits as coefficients. Interestingly, the high school texts my students had used in their practice teaching had not made the connection back to the standard algorithm for long division.
The complex plane

The algebraic topic that gene rated the most interest in the last two years arose from the following questions –

1. Can one introduce a multiplication on the vector space R2 so that the resulting structure is a field?
2. Can one do so in a way that contains a square root for -1?
3. Can one then define an order relation on this field satisfying the axioms for an ordered field?

The answers to Q1 and Q2 were essentially a construction of the complex numbers C. The answer to Q3 required some care to avoid the assumption that the order relation had to extend the usual order relation on R. The discussion also clarified the need to be very specific about what structure is meant when asking about a possible isomorphism.

Once we justified the definition
\[ e^{i\theta} = \cos(\theta) + i \sin(\theta) \]
and asserted that the ordinary rules for exponents carried over to complex exponents, the students were delighted to see that they could use these facts to make coherent many trig identities that they had previously seen as ‘arbitrary and capricious’.

Isomorphism in one structure but not another.

From “rule” to “isomorphism.”
Some concluding comments on the experience

The students in the Connections course range from fairly strong math majors to very weak ones. The weak students often had very strong commitment to a teaching career and has declined to take seriously material that they did not initially see as relevant to their professional development. Several of these weaker students told me that did not recall even hearing some of the basic theorems in analysis, e.g. the one we quote colloquially as “the limit of the sum is the sum of the limits”. Others seem to ignore the issue of quantifiers or domains of discourse in analyzing the validity of proposed “rules”. Such as

\[ A^2 + B^2 = C^2 \quad \text{or} \quad \sqrt{(A^2+B^2)} \leq A + B \]

In the first they simply assumed that the symbols had their conventional meanings from the Pythagorean Theorem and in the second they seemed to assume that the letters represented non-negative reals.

Even the stronger students were surprised to see connections that had not been pointed out by earlier professors. I suspect that many of my colleagues still believe what I used to believe – namely that these connections are so obvious or so trivial that they need not be mentioned. I hope that my materials for the Connections course may help my colleagues enhance the education they offer their students, especially those not headed for Ph.D. programs in pure mathematics. Come to think of it, many even future pure mathematicians ought to contemplate the uses of mathematics in teaching as well as in other applications.

Ph.D granting depts prepare Ph.D faculty for depts
who whose major mission is educating teachers

THNX QUESTIONS

DEPT SUPPORT

REQUIREMENT

TOPIC LIST
Connecting
Advanced Mathematics
To
High School Content

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joint work with

Lara Alcock, Essex, UK

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Connecting Mathematical Reasoning to HS Algebra

Problem 1.
Solve $x^2 - 2x - 15 = 0$
How many solutions?
Check them by substitution

Problem 2
Repeat with $u - 3\sqrt{u} - 28 = 0$
How many solutions?
Check?
Did you make algebra errors?
Task: Solve \( x - \sqrt{x} - 6 = 0 \)

Work:
\[
(\sqrt{x} - 3)(\sqrt{x} + 2) = 0
\]
\[
\sqrt{x} - 3 = 0 \quad \sqrt{x} + 2 = 0
\]
\[
\sqrt{x} = 3 \quad \sqrt{x} = -2
\]
\[
x = 9 \quad x = 4
\]

Answer: \( (9, 4) \)

Check:
\[
9 - \sqrt{9} - 6 = 9 - 3 - 6 = 0 \quad \checkmark
\]
\[
4 - \sqrt{4} - 6 = 4 - 2 - 6 = -4 \quad \times
\]

Revised Answer: \( 9 \)
**Task** Solve \( X - \sqrt{X} - 6 = 0 \) (*)

**Work**

Assume

1. \( X \in \mathbb{R} \) and \( X - \sqrt{X} - 6 = 0 \)
2. \((\sqrt{X} - 3)(\sqrt{X} + 2) = X - \sqrt{X} - 6\)
3. By (1) and (2) we get \((\sqrt{X} - 3)(\sqrt{X} + 2) = 0\)
4. By zero product property \(\sqrt{X} - 3 = 0 \) or \( \sqrt{X} + 2 = 0 \)
5. By addition \(\sqrt{X} = 3 \) or \( \sqrt{X} = -2 \)
6. By squaring \( X = 9 \) or \( X = 4 \)

**Check:** 
- \(9 - \sqrt{9} - 6 = 9 - 3 - 6 = 0\)
- \(4 - \sqrt{4} - 6 = 4 - 2 - 6 = -4 \neq 0\)

**Result** \((1) \iff X = 9\)

The only real solution of (*) is 9
Consider the expression
\[ E = \frac{0.\overline{9}}{9} \ldots \]
all digits equal 9

What should \( E \) mean?

\[ E \geq \text{each truncation} \]

\[ E \geq \frac{0.\overline{9}}{9} = 1 - \left(\frac{1}{10}\right)^n \]
\( n \) digits \( \cdot \) each = 9

\[ E \leq 1 \]

\( E \) is one upper bound
for the truncations

\( E \) is the least upper
bound for them

\( E \) cannot be less than 1

\( E \) must equal 1, at least
if it has any sense at all.
Connecting Limits and Continuity to HS Issues

Problem 1
What does it mean to say “Number A approximates Quantity Q to \( n \) decimal places”?
Problem 2

Measure the length and width of a rectangle.

a. How closely does the product of the measurements

\[ L_{\text{meas}} \times W_{\text{meas}} \]

approximate the true area

\[ A = L_{\text{true}} \times W_{\text{true}} \]?

b. What about approximating perimeter?

c. How do answers depend on magnitude of measured quantities?
H_m and W_m are measured. We learn

\[ 11 < H_m < 12, \quad 73 < W_m < 74, \]

\[ |H - H_m| < \varepsilon, \quad |W - W_m| < 3. \]

How big is

\[ |HW - H_m W_m| \]

in terms of what we know?

How small must \( \varepsilon \) be to ensure

\[ |HW - H_m W_m| < 10^{-5} ? \]
\( \langle \mathbb{R}^2, \oplus \rangle \) is n.r. sp. over \( \mathbb{R} \)

Basis vectors \( u = (1, 0) \), \( v = (0, 1) \)

Can we define multiplication \( \otimes \)

So \( \langle \mathbb{R}^2, \oplus, \otimes \rangle \) is a field

& \( u \) is the mult. ident.

& \( u \otimes u = -u \) ?

Yes, \( \langle \mathbb{R}^2, \oplus, \otimes \rangle = \mathbb{C}, \ i = u = \sqrt{-1} \).

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Can we define an order rel \( < \) so \( \langle \mathbb{C}, < \rangle \) is an ordered field?

NO

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In \( \mathbb{C} \)

\( \langle \{ s \in \mathbb{R} \} \cup \ominus : \oplus \rangle \cong \langle \mathbb{R}, + \rangle \) as Groups

\( \langle \{ s \in \mathbb{R} \} \cup \ominus, \oplus, \otimes \rangle \not\cong \langle \mathbb{R}, +, \cdot \rangle \) as Fields

\( \langle \{ s \in \mathbb{C} \} \cup \ominus, \oplus, \otimes \rangle \cong \langle \mathbb{R}, +, \cdot \rangle \) as Fields
Results of the advanced mathematics test reveal some unexpected weaknesses. Despite the fact that about one-quarter of the test related to calculus and that one-half of the U.S. advanced mathematics students were actually studying calculus, it was in geometry, not calculus, where U.S. students performed worst. This is consistent with performance in grades 4 and 8, but unexpected because these advanced students have all had formal geometry coursework. The results show that both geometry and algebra need to be key subjects of study throughout the curriculum.

Neal Lane, Director
National Science Foundation
One important use of assessment is to find out where education has been weak. As an example, consider the results in geometry in the TIMSS-95 assessment of knowledge of advanced students in the last year of high school, [7]. The average score for all 17 of the countries was 500. The scores ranged from 424 to 548. The lowest four average scores with the percent of students sampled were

<table>
<thead>
<tr>
<th>Country</th>
<th>Score</th>
<th>Percent of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>USA</td>
<td>424</td>
<td>14</td>
</tr>
<tr>
<td>Austria</td>
<td>462</td>
<td>33</td>
</tr>
<tr>
<td>Slovenia</td>
<td>476</td>
<td>75</td>
</tr>
<tr>
<td>Italy</td>
<td>480</td>
<td>14</td>
</tr>
</tbody>
</table>

To illustrate some of the problems in the knowledge of geometry, consider a classic simple problem from synthetic geometry.

In the $\triangle ABC$ the altitudes $BN$ and $CM$ intersect at point $S$. The measure of $\angle MSB$ is $40^\circ$ and the measure of $\angle SBC$ is $20^\circ$. Write a PROOF of the following statement:

"$\triangle ABC$ is isosceles."

Give geometric reasons for statements in your proof.

All that is needed to solve this problem is to know that the sum of the angles of a triangle is 180 degrees; and write that angle BSC is 140 degrees, so angle SCB is 20 degrees. Angle CSN is 40 degrees, so angles MBS and NCS are each 50 degrees. Thus the base angles are equal, so the triangle is isosceles. This is an easy example of a type of problem which is given in Grade 8 in Hong Kong, Japan and Singapore. None of these East Asian countries participated in this part of TIMSS-95.

The international average of correct responses was not high, only 35%. The range was from 10% to 65% with the U.S. score the lowest at 10% correct.
K18. In the ΔABC the altitudes BN and CM intersect at point S. The measure of ∠MSB is 40° and the measure of ∠SBC is 20°. Write a PROOF of the following statement:

"ΔABC is isosceles."

Give geometric reasons for statements in your proof.
Using TIMSS Videos to Improve Learning of Mathematics: A Resource Guide

Created to organize and provide guided access to the TIMSS video lessons and their many valuable insights for mathematics educators.

Richard Askey and Patsy Wang-Iverson, Editors

Introduction

Abbreviations Used in the Guide

How to Use This Guide

Lesson Resources

Lessons by Country. A complete list of lessons with links to all related resources.

Lessons by Topic. Lessons indexed by mathematical and pedagogical topic.

Lessons with Detailed Description and Commentary. Fourteen lessons with entry points for detailed discussion.

Lessons Treating algebra, Pythagorean theorem, and proof.
Other Resources

An Overview of Mathematical and Pedagogical Issues in the Lessons

Terminology Used in the Guide

Using Technology

Using Video Lessons for Professional Development

Suggestions for Using These Lessons with Teachers

Background of the TIMSS 1999 Video Study

TIMSS Public Release Lesson CDs

Acknowledgments
Hong Kong geometry lesson on polygons

Near the start of this lesson students were asked what a regular polygon is.

Next, they were asked to draw a quadrilateral which is equilateral but not equiangular. They were also asked to draw a quadrilateral which is equiangular but not equilateral.

Then they were asked to draw the same two types of figures for a pentagon.
Ex. 4. Given $AB \parallel CD$. Prove $\angle b = \angle a + \angle c$.

Ex. 5. Conversely, given $\angle b = \angle a + \angle c$. Prove $AB \parallel CD$.

Some of the principal auxiliary lines used on rectilinear figures are:

- A line connecting two given points.
- A line through a given point parallel to a given line.
- A line through a given point perpendicular to a given line.
- A line making a given angle with a given line.
- A line produced its own length.
178. Use of auxiliary lines. — The demonstration of a property of a geometrical figure is frequently facilitated by drawing one or more auxiliary lines on the figure. For examples of the use of such lines, see Props. IV, V, IX, etc., of Book One.

Some of the principal auxiliary lines used on rectilinear figures are:

- A line connecting two given points.
- A line through a given point parallel to a given line.
- A line through a given point perpendicular to a given line.
- A line making a given angle with a given line.
- A line produced its own length.

**Auxiliary Lines**

1. In the quadrilateral $ABCD$, given $AB = AD$, and $BC = CD$. Prove $\angle B = \angle D$.
   [Sug. Draw $AC$, etc.]

2. Prove that the angles at the base of an isosceles trapezoid are equal.

3. State and prove the converse of Ex. 2.

4. Given $AB \parallel CD$. Prove $\angle b = \angle a + \angle c$.

5. Conversely, given $\angle b = \angle a + \angle c$. Prove $AB \parallel CD$.

6. The median to the hypotenuse of a right triangle is one half the hypotenuse.
   [Sug. Draw $DF \parallel BC$. Then $AF = FC$ (§ 169), etc.]

7. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.
   [Sug. Draw the median to the hypotenuse, etc.]

8. In an isosceles triangle, the sum of the perpendiculars drawn from any point in the base to the legs is equal to the altitude upon one of the legs.

   In some cases, it is useful to draw two or more auxiliary lines.

9. If the opposite sides of a hexagon are equal and one pair of sides ($AB$ and $CD$) are parallel, the opposite angles of the hexagon are equal.
   [Sug. Draw $AC$ and $BD$; use § 101.]
**Ex. 4.** In the trusses of steel bridges, why are the beams and rods arranged, as far as possible, so as to form a network of triangles and not of quadrilaterals, or pentagons, for instance? (See Ex. 19, p. 88; also § 87.)

**Ex. 5.** Let $DC$ and $FC$ be two walls perpendicular to the plane of the paper. Let small mirrors be attached to these walls at $A$ and $B$. Let $\angle C = 45^\circ$. Let a ray of light pass through $Q$ to $A$, and be reflected to $B$ and thence to $P$. Prove that $\angle APB$ is a right angle.

[Sug. Use the principle for the angles of incidence and reflection stated in Ex. 2.

Then in the triangle $ABC$, $x+y+45^\circ = 180^\circ$, or $x+y = 135^\circ$.

About the points $A$ and $B$; $2x+a+2y+b = 360^\circ$.

$\therefore a+b = 90^\circ$, etc.]

The preceding is the principle of an instrument, called the optical square, used by foresters in constructing right angles. A ray of light coming from $R$ through a small hole at $B$ above the mirror will make a right angle with the ray coming from $Q$ through $P$ to $A$.

**Ex. 6.** By the aid of squared paper, construct the following designs. Extend the last design to include at least five major loops.
**Activity 2**

**Vertex Inside Circle—Two Secants (Case 2)**

1. \( \angle AVC \) is an exterior angle of \( \triangle ADV \). What is the relationship between the measure of \( \angle AVC \) and the measures of \( \angle 1 \) and \( \angle 2 \)?

2. Copy and complete the following table:

<table>
<thead>
<tr>
<th>m( \widehat{AC} )</th>
<th>m( \widehat{BD} )</th>
<th>m( \angle 1 )</th>
<th>m( \angle 2 )</th>
<th>m( \angle AVC )</th>
<th>m( \angle DVB )</th>
</tr>
</thead>
<tbody>
<tr>
<td>160°</td>
<td>40°</td>
<td>80°</td>
<td>20°</td>
<td>100°</td>
<td>100°</td>
</tr>
<tr>
<td>180°</td>
<td>70°</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>200°</td>
<td>60°</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

3. Based on your results, complete the theorem below.

**Theorem**

The measure of an angle formed by two secants or chords that intersect in the interior of a circle is \( \_\_ \) the \( \_\_ \) of the measures of the arcs intercepted by the angle and its vertical angle. 9.4.2

**Activity 3**

**Vertex Outside Circle—Two Secants (Case 3b)**

1. \( \angle 1 \) is an exterior angle of \( \triangle BVC \). What is the relationship between the measure of \( \angle 1 \) and the measures of \( \angle 2 \) and \( \angle AVC \)?

2. Copy and complete the following table:

<table>
<thead>
<tr>
<th>m( \widehat{BD} )</th>
<th>m( \widehat{AC} )</th>
<th>m( \angle 1 )</th>
<th>m( \angle 2 )</th>
<th>m( \angle AVC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>200°</td>
<td>40°</td>
<td>100°</td>
<td>20°</td>
<td>80°</td>
</tr>
<tr>
<td>250°</td>
<td>60°</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>100°</td>
<td>50°</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

3. Based on your results, complete the theorem below.

**Theorem**

The measure of an angle formed by two secants that intersect in the exterior of a circle is \( \_\_ \) the \( \_\_ \) of the measures of the intercepted arcs. 9.4.3

You will explore cases 3a and 3c in Exercises 27–37.
9. $BCD$ and $ECF$ are straight lines.  
10. $AB$ and $DE$ are straight lines.

11. $AD$ and $BE$ intersect at $C$.

12. $B$ and $D$ are connected by $C$.

13. $W$ intersects $Z$ at $q$.

14. $B$ and $A$ are connected by $E$.

15. $AE$ and $BC$ are straight lines.

16. $A$ and $B$ are connected by $C$.  

$AE$ and $BC$ are straight lines.
3. Find the unknowns in the following figures.

(a) $ADC$ and $BED$ are straight lines.
An explanation like the one on pages 118–119 using the conditions for congruent triangles is also a proof.

Problem 2
Prove that

$$\angle ADC = \angle A + \angle B + \angle C$$

in the diagram to the right, assuming that $DE$ is an extension of $BD$. Think about other ways of proving this.

Problem 3
In diagrams (1) and (2) below, the sum of the five marked angles is $180^\circ$. Prove this from what we learned on the preceding page.

(1) \hspace{5cm} (2)

Now let’s start proving various properties of figures.
15. Prove that $a+b+c+d+e = 180^\circ$ in the figure.

**Proof:**

In $\triangle CFE$,

$$f = \_\_\_\_\_\_\_$$

In $\triangle ADG$,

$$g = \_\_\_\_\_\_\_$$

In $\triangle BFG$,

$$b + f + g = \_\_\_\_\_\_\_$$

$$\therefore \quad b + \_\_\_\_ + \_\_\_\_ = \_\_\_\_\_\_\_$$

i.e. $a+b+c+d+e = 180^\circ$
6. In the figure, $AIJC$, $CKLE$, $EMNG$, $GHIB$, $BJKD$, $DLMF$ and $FNHA$ are straight lines, find the sum of $a$, $b$, $c$, $d$, $e$, $f$ and $g$.

7. Construct a regular 12-gon of sides 3 cm with a ruler and a protractor.
Connections

15. A regular decagon and a star are shown below. Measure the angles inside the star to find the angle sum of the star. Compare your results to the angle sum for a regular decagon.

Connections

15. The star has five interior angles that are $35^\circ$ and five that are $253^\circ$. Adding one of each of those gives $288^\circ$, which is the same as adding two of the regular decagon's angles ($144^\circ + 144^\circ$). The star has the same angle sum as the regular decagon.
Eda and Azusa each own a piece of land that lies between the same pair of lines. Their common boundary is formed by a bent line segment as shown. The problem is to change the bent line into a straight line segment that still divides the region into two pieces, each with the same area as before.

Figure 2

Despite the previous review, the problem is still going to be a challenge for eighth graders, and it is fair to infer that the teacher understands this very well. In geometry, one of the most difficult challenges in a construction or proof is determining where to put the auxiliary lines. These lines are needed to construct the angles, parallel lines, triangle(s), and so on that must be present before a geometry theorem or principle can be applied to solve the problem. For the exercise in Figure 2, the key step is to draw two crucial auxiliary lines. One defines the base of a triangle that must be transformed in a way that preserves its area. The other is parallel to this base, and runs through its opposing vertex.

So what should a master instructor do? The answer is on the tape.

After explaining the problem, the teacher asks the students to estimate where the solution line should go, playfully places his pointer in various positions that begin in obviously incorrect locations and progresses toward more plausible replacements for the bent line. Now here is the point. With the exception of two positionings over a duration of about one second (which come shortly after the frame shown in Figure 4), none of his trial placements approximate either of the two answers that are the only solutions any student will find.
For completeness, we show the two ways that the triangle transformation technique can be used to solve the problem. In order to make the connection between the review material and the follow-up Eda-Azusa exercise absolutely clear, the solution with its two versions have been rotated to present the same perspective as in Figure 6.
Two nonvertical lines are parallel if and only if their slopes are equal. Two nonvertical lines are perpendicular if and only if the product of their slopes is \(-1\).

\[(a - b)^2 + \frac{1}{2}ab \times 4 = c^2 \]

\[a^2 + b^2 = c^2\]
Proof of the Pythagorean Theorem
Questions for prospective high school teachers in Michigan around 1900

Construct a square equivalent to 2/3 of a given hexagon.

For a cyclic quadrilateral, prove that the product of the diagonals is the sum of the products of the opposite sides.

Demonstrate that the volume of a pyramid is one-third the product of the area of the base and the height.

Demonstrate that the medians of a triangle meet at a trisection point.
I.2. Theorem. The three medians of a triangle intersect at one point; this point cuts a third part of each median measured from the corresponding side.

In $\triangle ABC$ (Figure 127), take any two medians $AE$ and $BD$, intersecting at a point $O$, and prove that

$$\frac{OD}{\frac{1}{3} \cdot AE} = \frac{OD}{\frac{1}{3} \cdot BD},$$

and $OD = \frac{1}{3} \cdot AB$.

For this, project $O$ and $O'$ to the points $A$ and $C$ and consider the parallel bisector $DE$ of $\triangle ABO$, then $DE \parallel AA'$ and $DF \parallel AA'$. The segment of two sides of $\triangle ABO$, then $DF \parallel AA'$ and $DE \parallel AA'$ connects the midpoints of two sides of $\triangle ABO$, and hence $DE$ connects the midpoints of two sides of $\triangle ABC$, and hence $DE || AA'$ and $DF \parallel AA'$. From this, we conclude that $DF || AA'$ and $DE \parallel EE'$. If $\alpha$ follows that $O' = O$ and $OD = \frac{1}{3} \cdot AB$, i.e., that $O$ is the center of $\triangle ABC$.

If we consider from the third median and one of the medians $AA'$ $O'$, then we similarly find that their intersection point cuts from $O$ of each of them a third part measured from the foot. Therefore the third median must intersect the medians $AE$ and $BD$ at the very same point $O$.

Remark. (1) It is known from physics that the intersection point of the medians of a triangle is the center of the center of mass or centroid of the triangle. It is also called the center of gravity; it always lies inside the triangle.

(2) If three (or more) lines intersect at one point, the called concurrent. Thus we can say that the orthocenter, circumcenter, incenter and the center of gravity of a triangle are concurrent points of its subject's medians, angle bisectors, and perpendicular bisectors of its sides respectively.
The Centroid of a Triangle

A line segment that connects a vertex of a triangle with the midpoint of the opposite side is called a median.

**Problem 1** Draw a triangle, and then draw all its medians.

If two medians $AL$ and $BM$ of $\triangle ABC$ intersect at point $G$, then

$$AG : GL = 2 : 1$$

**Problem 2** Assume that a straight line passing through $L$ parallel to $BM$ intersects $AC$ at $D$. Now prove

(1) $MD = DC$  
(2) $AM : MD = 2 : 1$

and use those facts to demonstrate that $AG : GL = 2 : 1$.

Next, if we assume that two medians $AL$ and $CN$ of $\triangle ABC$ intersect at $H$, we can show that

$$AH : HL = 2 : 1$$

by the same procedure as in Problem 2.

Since both $G$ and $H$ divide median $AL$ into parts with a ratio of $2 : 1$, $G$ and $H$ must coincide. In other words, the three medians $AL$, $BM$, and $CN$ all intersect at point $G$, which divides median $AL$ into parts with a ratio of $2 : 1$.

**Centroid Theorem**

Theorem: The three medians of a triangle intersect at a single point. That intersection divides the medians into parts with a ratio of $2 : 1$. 
Theorem. The product of the diagonals of an inscribed quadrilateral is equal to the sum of the products of its opposite sides.

This proposition is called Ptolemy's theorem after a Greek astronomer Claudius Ptolemy (85 – 165 A.D.) who discovered it.

Let \( AC \) and \( BD \) be the diagonals of an inscribed quadrilateral \( ABCD \) (Figure 198). It is required to prove that

\[
AC \cdot BD = AB \cdot CD + BC \cdot AD.
\]

Construct the angle \( BAE \) congruent to \( \angle DAC \), and let \( E \) be the intersection point of the side \( AE \) of this angle with the diagonal \( BD \). The triangles \( ABE \) and \( ADC \) (shaded in Figure 198) are similar, since their angles \( B \) and \( C \) are congruent (as inscribed intercepting the same arc \( AD \)), and the angles at the common vertex \( A \) are congruent by construction. From the similarity, we find:

\[
AB : AC = BE : CD, \quad \text{i.e.} \quad AC \cdot BE = AB \cdot CD.
\]

Consider now another pair of triangles, namely \( \triangle ABC \) and \( \triangle AED \) (shaded in Figure 199). They are similar, since their angles \( BAC \) and \( DAE \) are congruent (as supplementing to \( \angle BAD \) the angle congruent by construction), and the angles \( ACB \) and \( ADB \) are congruent as inscribed intercepting the same angle \( AB \). We obtain:

\[
BC : ED = AC : AD, \quad \text{i.e.} \quad AC \cdot ED = BC \cdot AD.
\]

Summing the two equality, we find:

\[
AC(BE + ED) = AB \cdot CD + BC \cdot AD, \quad \text{where} \quad BE + ED = BD.
\]
EXERCISES

320. Construct a triangle, given its base and two medians drawn from the endpoints of the base.

321. Construct a triangle, given its three medians.

322. Into a given circle, inscribe a triangle such that the extensions of its angle bisectors intersect the circle at three given points.

323. Into a given circle, inscribe a triangle such that the extensions of its altitudes intersect the circle at three given points.

324.* Construct a triangle given its circumscribed circle and the three points on it at which the altitude, the angle bisector and the median, drawn from the same vertex, intersect the circle.

325.* Prove that connecting the feet of the altitudes of a given triangle, we obtain another triangle for which the altitudes of the given triangle are angle bisectors.

326.* Prove that the barycenter of a triangle lies on the line segment connecting the circumcenter and the orthocenter, and that it cuts a third part of this segment measured from the circumcenter.

Remark: This segment is called Euler’s line of the triangle.

327.* Prove that for every triangle, the following nine points lie on the same circle (called Euler’s circle, or the nine-point circle of the triangle): three midpoints of the sides, three feet of the altitudes, and three midpoints of the segments connecting the orthocenter with the vertices of the triangle.

328.* Prove that for every triangle, the center of Euler’s circle lies on Euler’s line and bisects it.

Remark: Moreover, according to Feuerbach’s theorem, for every triangle, the nine-point circle is tangent to the inscribed and all three exscribed circles.
EXAMINATION A
EDUCATION DIVISION:
ELEMENTARY, SECONDARY,
INFORMATION SCIENCES

Time: 90 minutes  
Subjects: Math I, Algebra, Geometry, Basic Analysis  
Number of exam takers: 1438  
Number of those accepted: 368

1. Let $\alpha, \beta$ be two solutions of the second-degree equation 
   \[ x^2 - px + 1 = 0. \]

   Let $\alpha', \beta'$ be two solutions of the second-degree equation 
   \[ z^2 - z + q = 0. \]

   Express 
   \[ (\alpha' - \alpha)(\alpha' - \beta)(\beta' - \alpha)(\beta' - \beta) \]

   in terms of $p, q$.

2. Let 
   \[ A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ 1 & -2 \end{pmatrix}. \]

   If $(A + B)^2 = A^2 + 2AB + B^2$, determine the values of $a$ and $b$.

3. Graph the region bounded by the simultaneous inequalities 
   \[ x - 2y^2 \geq 0 \]
   \[ 1 - x - |y| \geq 0. \]

   Find the area of this region.

4. Sequence $\{a_n\}$ satisfies 
   \[ a_1 + 2a_2 + 3a_3 + \cdots + na_n = \frac{n+1}{n+2} \quad (n \geq 1). \]

   Find the sum 
   \[ S_n = a_1 + a_2 + a_3 + \cdots + a_n. \]

5. Let $a, b, c$ be the sides of $\triangle ABC$ and $S$ be the area. From an interior point $P$, draw the line perpendicular to each side with lengths $x$, $y$, and $z$, respectively, as is shown below:

(1) Express $S$ in terms of $a, b, c, x, y, \text{ and } z$.
(2) Let $P'(x, y, z)$ be a point corresponding to point $P$. Prove that as $P$ moves within $\triangle ABC$, the graph of $P'(x, y, z)$ describes a triangle.
(3) Prove that if $P$ is the center of gravity of $\triangle ABC$, then $P'$ is also the center of gravity of the triangle obtained in (2).